

Discrete-Time Goldfishing^{*}

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Abstract. The original *continuous-time* “goldfish” dynamical system is characterized by two neat formulas, the first of which provides the N Newtonian equations of motion of this dynamical system, while the second provides the solution of the corresponding initial-value problem. Several other, more general, *solvable* dynamical systems “of goldfish type” have been identified over time, featuring, in the right-hand (“forces”) side of their Newtonian equations of motion, in addition to other contributions, a velocity-dependent term such as that appearing in the right-hand side of the first formula mentioned above. The *solvable* character of these models allows detailed analyses of their behavior, which in some cases is quite remarkable (for instance *isochronous* or *asymptotically isochronous*). In this paper we introduce and discuss various *discrete-time* dynamical systems, which are as well *solvable*, which also display interesting behaviors (including *isochrony* and *asymptotic isochrony*) and which reduce to dynamical systems of goldfish type in the limit when the *discrete-time* independent variable $\ell = 0, 1, 2, \dots$ becomes the standard *continuous-time* independent variable t , $0 \leq t < \infty$.

Key words: nonlinear discrete-time dynamical systems; integrable and solvable maps; isochronous discrete-time dynamical systems; discrete-time dynamical systems of goldfish type

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1 Introduction

The original “goldfish” dynamical system [1, 2] is characterized by the system of N Newtonian equations of motion

$$\ddot{z}_n = \sum_{k=1, k \neq n}^N \frac{2\dot{z}_n \dot{z}_k}{z_n - z_k}, \quad n = 1, \dots, N, \quad (1.1a)$$

and by the following neat prescription yielding the solution of the corresponding initial-value problem: the N values of the dependent variables $z_n \equiv z_n(t)$ at time t are the N solutions of the algebraic equation (for the unknown z)

$$\sum_{k=1}^N \frac{\dot{z}_k(0)}{z - z_k(0)} = \frac{1}{t}, \quad (1.1b)$$

i.e. the N roots of the polynomial equation of degree N in the variable z that obtains by multiplying this formula by the polynomial $\prod_{j=1}^N [z - z_j(0)]$.

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Notation 1.1. Here and hereafter N is an arbitrary *positive integer* (generally $N \geq 2$), superimposed dots denote differentiations with respect to the independent variable t (“continuous time”), and the N dependent variables $z_n \equiv z_n(t)$ may be interpreted as the coordinates of N point-like unit-mass particles – hence \dot{z}_n denote their velocities and \ddot{z}_n their accelerations, consistently with the interpretation of (1.1a) as a set of Newtonian equations of motion with velocity-dependent forces. The indices – such as n, m, j, k – generally run from 1 to N (below, as a convenient reminder, we often indicate this explicitly; as well as the exceptions to this rule).

Hereafter we denote as “dynamical system of goldfish type” any dynamical system characterized by Newtonian equations of motion featuring in their right-hand sides – which have, in the Newtonian context, the significance of “forces” – a velocity-dependent term such as that appearing in the right-hand side of (1.1a) (of course in addition to other terms). Let us also emphasize that the dynamical system (1.1) is the simplest model belonging to the Ruijsenaars–Schneider *integrable* class [3, 4].

For instance a simple extension of the above model (reducing to it for $\omega = 0$) is characterized by the Newtonian equations of motion:

$$\ddot{z}_n = (1 - 2\alpha)i\omega\dot{z}_n + \alpha(\alpha - 1)\omega^2 z_n + \sum_{m=1, m \neq n}^N \frac{2(\dot{z}_n + i\alpha\omega z_n)(\dot{z}_m + i\alpha\omega z_m)}{z_n - z_m}, \quad n = 1, \dots, N. \quad (1.2a)$$

The corresponding solution of the initial-value problem is again given by a simple rule: the N values of the dependent variables $z_n \equiv z_n(t)$ at time t are related by the formula

$$z_n(t) = \zeta_n(t) \exp(-i\alpha\omega t), \quad n = 1, \dots, N, \quad (1.2b)$$

to the N solutions $\zeta_n(t)$ of the algebraic equation (for the unknown ζ)

$$\sum_{k=1}^N \frac{\dot{z}_k(0) + i\alpha\omega z_k(0)}{\zeta - z_k(0)} = \frac{i\omega}{\exp(i\omega t) - 1} \quad (1.2c)$$

(which again gets transformed into a polynomial equation of degree N in ζ after multiplication by the polynomial $\prod_{j=1}^N [\zeta - z_j(0)]$); or, equivalently but more directly, the N values of the dependent variables $z_n \equiv z_n(t)$ at time t are the N solutions $z_n(t)$ of the algebraic equation (for the unknown z)

$$\sum_{k=1}^N \frac{\dot{z}_k(0) + i\alpha\omega z_k(0)}{z - z_k(0) \exp(-i\alpha\omega t)} = \frac{i\omega \exp(i\alpha\omega t)}{\exp(i\omega t) - 1}. \quad (1.2d)$$

Notation 1.2. Here and hereafter i is the imaginary unit, $i^2 = -1$, ω is an arbitrary constant (dimensionally, an inverse time), and α is an arbitrary (dimensionless) constant. Clearly – unless both ω and $\alpha\omega$ are both imaginary, $\text{Re}(\omega) = \text{Re}(\alpha\omega) = 0$ – the time-evolution of this system takes place in the *complex* z -plane, i.e. the dependent variables $z_n \equiv z_n(t)$ are *complex*; but it may as well be viewed as describing the evolution of N point-like particles moving in the *real* xy -plane – whose positions at time t are characterized by the (*real*) Cartesian coordinates $x_n \equiv x_n(t)$, $y_n \equiv y_n(t)$ – by setting $z_n(t) = x_n(t) + iy_n(t)$; and one of the remarkable features of the resulting *real* model is the possibility to write its Newtonian equations of motion in *covariant* – i.e., *rotation-invariant* – form, see Chapter 4 of [4]. Hereafter we generally refer to this model, and its generalizations, see below, in their *complex* versions.

Remark 1.1. Clearly for $\omega = 0$ this model, (1.2), reduces to the previous model (1.1). For $\omega \neq 0$ the time evolution of this model, (1.2), depends mainly on the values of the two constants ω and $\alpha\omega$, as displayed by its solution, see (1.2d). If both these constants are *real*, $\text{Im}(\omega) = \text{Im}(\alpha\omega) = 0$ (hence as well $\text{Im}(\alpha) = 0$), the time evolution of this model is *confined*, indeed *completely periodic* if the *real* number α is *rational*, while if α is *irrational* it is *multiply periodic*, being a nonlinear superposition of two periodic evolutions with the two noncongruent periods $T = 2\pi/|\omega|$ and T/α . Note that these outcomes obtain for *generic* initial data: hence, for α *rational*, $\alpha = q/p$ with q and p *coprime integers* (and $p > 0$), this system is *isochronous*, its *generic* solutions being *completely periodic* with period pT – or possibly with a period which is a, generally small, *integer multiple* of pT : indeed, when the equation (1.2d) is itself periodic with period pT , the *unordered set* of its N roots is clearly periodic with the same period pT , but the periodicity of the time-evolution of each individual coordinate $z_n(t)$ may then be a, generally small, *integer multiple* of pT due to the possibility that different roots get exchanged through the time evolution (for a discussion of this phenomenology – including a justification of the assertion that the relevant integer multiple of pT is generally small – see [5]).

On the other hand, if ω is *real* but α is *imaginary*, say $\alpha\omega = i\gamma$ with γ *real* and *nonvanishing*, then clearly in the remote future – i.e., as $t \rightarrow \infty$, and up to relative corrections of order $\exp(-|\gamma|t)$ – all the N coordinates $z_n(t)$ tend to the origin, $z_n(\infty) = 0$, if $\gamma < 0$, while if $\gamma > 0$ they all diverge (see (1.2b) and (1.2c)).

If instead ω is *not real*, $\text{Im}(\omega) \neq 0$, then in the remote future (i.e., as $t \rightarrow \infty$, and up to relative corrections of order $\exp(-|\text{Im}(\omega)|t)$) the N solutions of (1.2c) become asymptotically, if $\text{Im}(\omega) > 0$, the N solutions $\zeta_n = \zeta_n(\infty)$ of the *time-independent* polynomial equation of order N in ζ

$$\sum_{k=1}^N \frac{\dot{z}_k(0) + i\alpha\omega z_k(0)}{\zeta - z_k(0)} = -i\omega,$$

while if instead $\text{Im}(\omega) < 0$ the equation (1.2c) becomes, in the remote future, the *time-independent* polynomial equation

$$\sum_{k=1}^N \frac{\dot{z}_k(0) + i\alpha\omega z_k(0)}{\zeta - z_k(0)} = 0,$$

hence $N - 1$ of the solutions of (1.2c) tend asymptotically to the $N - 1$ solutions of this equation (polynomial of degree $N - 1$ in ζ) and one of them approaches asymptotically the *diverging* coordinate

$$\zeta_{\text{asy}}(t) = \exp(i\omega t) \sum_{k=1}^N [\dot{z}_k(0) + i\alpha\omega z_k(0)].$$

Note that this implies (see (1.2b)) that, if $\text{Im}(\omega) > 0$ but $\alpha\omega$ is *real*, $\alpha\omega = \rho$ with ρ *real* and *nonvanishing*, then the model (1.2) is *asymptotically isochronous*, its *generic* solutions becoming, in the remote future, *completely periodic* with period $2\pi/|\rho|$, up to corrections vanishing exponentially as $t \rightarrow \infty$ (for a more detailed discussion of the notion of *asymptotic isochrony* see Chapter 6, entitled “Asymptotically isochronous systems”, of [6]).

For $\omega = 0$ (i.e., when the model (1.2) reduces to (1.1)) it is possible to restrict consideration to *real* dependent variables z_n , but even then it is more interesting *not* to do so, so that the time evolution takes place in the plane rather than on the real line: see the remarkable behavior of this dynamical system in this case (“the game of musical chairs”), as detailed in Section 4.2.4 of [4]. Hence let us reiterate that we always consider the dependent variables z_n to be *complex*

numbers, both in the *continuous-time* case, $z_n \equiv z_n(t)$, $0 \leq t < \infty$, and (see below) in the *discrete-time* case, $z_n \equiv z_n(\ell)$, $\ell = 0, 1, 2, \dots$.

Another large class of *solvable* dynamical systems “of goldfish type” is characterized by the equations of motion

$$\begin{aligned} \ddot{z}_n = & a_1 \dot{z}_n + a_2 + a_3 z_n - 2(N-1)a_4 z_n^2 + \sum_{m=1, m \neq n}^N (z_n - z_m)^{-1} [2\dot{z}_n \dot{z}_m \\ & + (a_5 + a_6 z_n)(\dot{z}_n + \dot{z}_m) + a_7 z_n(\dot{z}_n z_m + \dot{z}_m z_n) + 2(a_8 + a_9 z_n + a_{10} z_n^2 + a_4 z_n^3)], \\ & n = 1, \dots, N, \end{aligned} \quad (1.3)$$

featuring 10 arbitrary constants (see [4, equation (2.3.3-2)]). In this case the *solvability* is achieved by identifying the N dependent variables $z_n(t)$ with the N roots of a time-dependent polynomial $\psi(z, t)$ of degree N in z satisfying a *linear* second-order PDE in the two independent variables z and t .

For an explanation of the origin of the name “goldfish” attributed to these models see Section 1.N of [6] and the literature cited there. In this book [6] (see in particular its Section 4.2.2, entitled “Goldfishing”, and the papers referred to there) several other *solvable* models “of goldfish type” are reported, including *isochronous* ones (i.e., models featuring solutions which are *completely periodic with a period independent of the initial data*). A few additional models of goldfish type have been identified more recently [7, 8, 9].

The most remarkable aspect of these dynamical systems is their *solvability*, namely the possibility to solve their initial-value problems by *algebraic* operations, amounting generally to finding the N eigenvalues of an $N \times N$ explicitly known time-dependent matrix (see below), or equivalently to finding the N roots of an explicitly known time-dependent polynomial of degree N (see for instance (1.1b) and (1.2d)). Quite interesting is also the identification of *multiply periodic*, *completely periodic*, or even *isochronous* or *asymptotically isochronous* cases.

In the present paper we present various *discrete-time* dynamical systems “of goldfish type”, so denoted because all these models reduce, in the limit when the *discrete-time* independent variable $\ell = 0, 1, 2, \dots$ becomes *continuous*, to *continuous-time* dynamical systems of goldfish type. All these models are moreover *solvable*, i.e. the solution of their initial value problems can be achieved by finding the N eigenvalues $z_n(\ell)$ of $N \times N$ matrices explicitly known in terms of the initial data and of the *discrete-time* independent variable ℓ ; or equivalently by finding the N roots $z_n(\ell)$ of a polynomial, of degree N in the complex variable z , as well explicitly known in terms of the initial data and of the *discrete-time* independent variable ℓ . Some of these models feature interesting behaviors, even *isochrony* or *asymptotic isochrony*. Two of these models (see Subsection 2.1 and 2.2) were treated in the paper [10], which has not been published because – after it was submitted for publication but before getting any feedback – new solvable models were identified and it was therefore considered preferable to report all these models in a single paper, this one. The main properties of each of these *discrete-time* models are reported in Section 2, and proven in Section 3. These properties include the display of the equations of motion of these *discrete-time* models, the solution of their initial-value problems, a terse discussion (for the first three models) of their behavior including the possibility that for special values of some of their parameters they possesses *periodic* or *multi-periodic* solutions or even display *isochrony* or *asymptotic isochrony*, and some mention of their *continuous-time* limits. Section 4 entitled “Outlook” concludes the paper: in it a general framework is outlined which might allow the identification of additional solvable *discrete-time* models. And some mathematical developments are confined to two appendices.

These findings are congruent with the recent surge of interest for *discrete-time* evolutions – in particular, such evolutions which are in some sense *integrable* or even *solvable*. Given the large body of research devoted to these topics over the last two decades, our reference to the

relevant literature shall be limited to citing the following surveys: [11, 12, 13, 14, 15]. But a special mention must be made of the seminal papers by Nijhoff, Ragnisco, Kuznetsov and Pang, see [16] as well as the earlier paper [17], where *discrete-time* versions were introduced of the well-known integrable “Ruijsenaars–Schneider” and “Calogero–Moser” dynamical systems; as well as of the paper by Suris [18], which treats specifically a *discrete-time* version of the original goldfish model. Some results of these papers refer to models whose equations of motion feature trigonometric/hyperbolic or even elliptic functions, and are therefore more general than those treated in the present paper, whose equations of motion only feature rational functions (see below); on the other hand the findings reported below include more general models than those previously treated, demonstrate the *solvability* of these models by an approach somewhat different from those previously employed, and, most significantly, display the possible emergence of remarkable phenomenologies – including *periodicity* and even *isochrony* or *asymptotic isochrony*, see below – not previously identified for this kind of *discrete-time* dynamical systems.

Let us also mention that the approach developed below also allows to identify and investigate *discrete-time* variants of another class of solvable *continuous-time* dynamical systems, the prototype of which is characterized by the Newtonian equations of motion

$$\ddot{z}_n = \sum_{k=1, k \neq n}^N \frac{c}{(z_n - z_k)^3}, \quad n = 1, \dots, N$$

(with c an arbitrary constant), instead of (1.1a). But in this paper we merely indicate, at the appropriate point, how to proceed in this direction, postponing a complete treatment of this development to a separate paper.

And let us finally pay tribute to Olshanetsky and Perelomov who were the first to show, more than 35 years ago, that the time-evolution of a nontrivial many-body system could be usefully identified with the evolution of the eigenvalues of a matrix itself evolving in a much simpler, explicitly solvable, manner: see [19] and their other papers referred to in Section 2.1.3.2 of [4], entitled “The technique of solution of Olshanetsky and Perelomov”. The present paper extends their approach to the *discrete-time* context.

2 Results

In this section we report the main results of this paper; they are then proven in Section 3.

Notation 2.1. Hereafter the dependent variables are indicated again as z_n , but they are now functions, $z_n \equiv z_n(\ell)$, of the *discrete-time* variable ℓ taking the *integer* values $\ell = 0, 1, 2, \dots$; and superimposed tildes indicate generally a *unit* increase of the independent variable ℓ , for instance $\tilde{z}_n \equiv z_n(\ell + 1)$, $\tilde{\tilde{z}}_n \equiv z_n(\ell + 2)$. Hereafter δ_{nm} is the standard Kronecker symbol, $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$, and underlined quantities are N -vectors, for instance $\underline{z} \equiv (z_1, \dots, z_N)$. For the remaining notation we refer to Notation 1.1, see above, and to specific indications given case-by-case below.

As reported in this section and explained in Sections 3 and 4, the solvable models considered in this paper generally feature three *equivalent* versions of the second-order “equations of motion” characterizing their evolution in *discrete-time*. The treatment of the first model given in Subsections 2.1 and 3.1 is somewhat more detailed than that provided for the other models in the subsequent subsections where, to avoid repetitions, we often refer to the treatments provided in Subsections 2.1 and 3.1. And already in this section, as well as in Section 3, we often take advantage – to simplify the presentation of some results – of *identities* and *lemmata* collected in Appendix A.

2.1 First model

The first model is defined by the following second-order *discrete-time* equations of motion: the N values of the twice-updated variables $\tilde{z}_n \equiv z_n(\ell + 2)$ are given, in terms of the $2N$ values of the variables $z_m \equiv z_m(\ell)$, $\tilde{z}_m \equiv z_m(\ell + 1)$, by the N roots of the following (*single*) algebraic equation in the unknown z ,

$$\sum_{k=1}^N \left[\left(\frac{\tilde{z}_k - az_k}{z - a\tilde{z}_k} \right) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{z}_k - az_j}{\tilde{z}_k - \tilde{z}_j} \right) \right] = 1, \quad (2.1a)$$

which clearly amounts to a polynomial equation of degree N in this variable z (as it is immediately seen by multiplying this equation by the polynomial $\prod_{m=1}^N (z - a\tilde{z}_m)$). Here and below a is an arbitrary (dimensionless, nonvanishing) constant. A neater version of this formula is easily obtained by multiplying it by a and by then using the identity (A.10) with $\eta_n = a\tilde{z}_n$, $\zeta_n = a^2 z_n$, $n = 1, \dots, N$. It reads

$$\prod_{j=1}^N \left(\frac{z - a^2 z_j}{z - a\tilde{z}_j} \right) = 1 + a. \quad (2.1b)$$

An *equivalent* formulation of this model is provided by the following system of N polynomial equations of degree N for the twice-updated coordinates $\tilde{z}_n \equiv z_n(\ell + 2)$:

$$\sum_{k=1}^N \left[\left(\frac{\tilde{z}_k - a\tilde{z}_k}{\tilde{z}_k - az_n} \right) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{z}_j - a\tilde{z}_k}{\tilde{z}_j - \tilde{z}_k} \right) \right] = a^{N-1}, \quad n = 1, \dots, N. \quad (2.2a)$$

Again, a neater version of this formula is easily obtained by dividing it by a and by then using the identity (A.10), now with $z = a^2 z_n$, $\eta_n = a\tilde{z}_n$, $\zeta_n = \tilde{z}_n$, $n = 1, \dots, N$. It reads

$$\prod_{j=1}^N \left(\frac{\tilde{z}_j - a^2 z_n}{\tilde{z}_j - a\tilde{z}_n} \right) = (1 + a)a^{N-1}, \quad n = 1, \dots, N. \quad (2.2b)$$

And a third, *equivalent* version of this model is provided by the following system of N polynomial equations of degree N for the twice-updated coordinates $\tilde{z}_n \equiv z_n(\ell + 2)$:

$$\prod_{j=1}^N \left(\frac{\tilde{z}_j - a\tilde{z}_n}{az_j - \tilde{z}_n} \right) = -a^{N-1}, \quad n = 1, \dots, N. \quad (2.3)$$

The similarities and differences among these three sets of “equations of motion”, (2.1), (2.2) and (2.3), are remarkable: let us reemphasize that they in fact yield the *same* evolution in discrete-time of the N coordinates $z_n \equiv z_n(\ell)$. Particularly remarkable is their similarity in the special $a = 1$ case, when the 3 versions (2.1b), (2.2b) and (2.3) of the equations of motion read as follows:

$$\begin{aligned} \prod_{j=1}^N \left(\frac{\tilde{z}_n - z_j}{\tilde{z}_n - \tilde{z}_j} \right) &= 2, \quad n = 1, \dots, N, \\ \prod_{j=1}^N \left(\frac{\tilde{z}_j - z_n}{\tilde{z}_j - z_n} \right) &= 2, \quad n = 1, \dots, N, \end{aligned}$$

$$\prod_{j=1, j \neq n}^N \left(\frac{\tilde{z}_j - \tilde{z}_n}{z_j - \tilde{z}_n} \right) = -1, \quad n = 1, \dots, N.$$

The last of these three systems coincides with equation (1.8) of [18].

Remark 2.1. This model, see (2.1), (2.2) and (2.3) – as the original goldfish model, see (1.2) – is *invariant* under an *arbitrary* rescaling of the dependent variables, $z_n \Rightarrow cz_n$ with c an *arbitrary constant*; including the special case $c = \exp(i\gamma)$ with γ an arbitrary *real* constant, corresponding to an overall *rotation* around the origin in the *complex* z -plane.

The solution of the initial-value problem for this model is given by the following

Proposition 2.1. *The N values $z_n(\ell)$ of the dependent variables at the discrete time ℓ are the N eigenvalues of the $N \times N$ matrix*

$$U_{nm}(\ell) = \delta_{nm} z_n(0) a^\ell + v_m(0) \frac{a^\ell - 1}{a - 1}, \quad n, m = 1, \dots, N \quad (2.4a)$$

with

$$v_m \equiv v_m(\underline{z}, \underline{\tilde{z}}) = \frac{\prod_{j=1}^N (\tilde{z}_j - a z_m)}{\prod_{j=1, j \neq m}^N [a(z_j - z_m)]}, \quad m = 1, \dots, N, \quad (2.4b)$$

where of course $v_m(0)$ indicates the value of $v_m(\underline{z}, \underline{\tilde{z}})$ corresponding to the initial data $\underline{z} = \underline{z}(0)$, $\underline{\tilde{z}} = \underline{\tilde{z}}(0) \equiv \underline{z}(1)$.

A neater, equivalent formulation of this finding – obtained from (2.4) via Lemma A.4 with $\zeta_n = z_n(0) a^\ell$ and $\eta_m = v_m(0)(a^\ell - 1)/(a - 1)$ – states that the N coordinates $z_n(\ell)$ are the N solutions of the following algebraic equation in z :

$$\sum_{k=1}^N \left\{ \left[\frac{z_k(1) - a z_k(0)}{z - a^\ell z_k(0)} \right] \prod_{j=1, j \neq k}^N \left[\frac{z_j(1) - a z_k(0)}{a z_j(0) - a z_k(0)} \right] \right\} = \frac{a - 1}{a^\ell - 1}.$$

And another, even neater, equivalent formulation – obtained from this via the identity (A.10) with z replaced by $z a^{1-\ell}$, $\eta_k = a z_k(0)$, $\zeta_j = \tilde{z}_j(0) = z_j(1)$ – states that the coordinates $z_n(\ell)$ are the N solutions of the following algebraic equation in z :

$$\prod_{k=1}^N \left[\frac{z - a^{\ell-1} z_k(1)}{z - a^\ell z_k(0)} \right] = \frac{a^{\ell-1} - 1}{a^\ell - 1}. \quad (2.5)$$

The last two equations become of course polynomial equations of degree N in z after multiplication by the product $\prod_{j=1}^N [z - a^\ell z_j(0)]$.

These formulas are also valid for $a = 1$ (by taking the obvious limit, i.e. replacing $(a^\ell - 1)/(a - 1)$ with ℓ). If instead $|a| < 1$, then clearly for all (*positive*) values of ℓ the matrix $U(\ell)$ is bounded and $U_{nm}(\infty) = v_m(0)/(1 - a)$; hence for all values of ℓ the N coordinates $z_n(\ell)$ are bounded and $N - 1$ of them vanish as $\ell \rightarrow \infty$ while one of them tends to the value

$$z_{\text{asy}} = (1 - a)^{-1} \sum_{k=1}^N v_k(0) = \frac{1}{1 - a} \sum_{k=1}^N [z_k(1) - a z_k(0)],$$

see (2.4b) and the identity (A.11) (with $\eta_k = a z_k(0)$, $\zeta_j = \tilde{z}_j(0) = z_j(1)$). If $a \neq 1$ but it has *unit* modulus,

$$a = \exp(2\pi i \lambda) \quad (2.6)$$

with λ *real* and *not integer*, then clearly the matrix $U(\ell)$ is again, for all values of ℓ , bounded; and if moreover λ is a (strictly, i.e. non integer) *rational* number,

$$\lambda = \frac{K}{L}, \quad a = \exp\left(\frac{2\pi i K}{L}\right), \quad (2.7)$$

with K and L two *coprime integers* and $L > 1$, then clearly the matrix $U(\ell)$ is *periodic* with period L ,

$$U(l + L) = U(l),$$

hence the (unordered) set of its N eigenvalues $z_n(\ell)$ is as well *periodic* with period L . This shows that in this case, see (2.7), the *discrete-time* goldfish model, see (2.1) or (2.2) or (2.3), is *isochronous*. On the other hand if λ is *real* and *irrational*, then clearly the time evolution of this *discrete-time* dynamical system is *not periodic*: indeed, while the right-hand side of (2.4a) (with (2.6) and λ *real* and *irrational*) is periodic (with *unit* period) as a function of the *real* variable $\tau = \lambda\ell$, clearly it is *not* periodic as a function of the variable ℓ taking the integer values $\ell = 0, 1, 2, \dots$. (We made this analysis, for convenience, referring to the coordinates $z_n(\ell)$ as the eigenvalues of $U(\ell)$, see (2.4); of course an analogous discussion could be made on the basis of the alternative identification of the coordinates $z_n(\ell)$ as the N roots of the polynomial equation (2.5) – whose similarity with (1.2d) is in any case to be noted, see below.)

To explore the transition from the *discrete-time* independent variable ℓ to the *continuous-time* variable t one makes the formal replacements

$$\ell \Rightarrow \frac{t}{\varepsilon}, \quad \ell + 1 \Rightarrow \frac{t + \varepsilon}{\varepsilon}, \quad \ell + 2 \Rightarrow \frac{t + 2\varepsilon}{\varepsilon}, \quad (2.8a)$$

$$a \Rightarrow 1 - i\omega\varepsilon, \quad (2.8b)$$

and (with a slight abuse of notation)

$$z_n(\ell) \Rightarrow z_n(t),$$

$$\begin{aligned} \tilde{z}_n(\ell) \equiv z_n(\ell + 1) &\Rightarrow z_n(t) + \varepsilon \dot{z}_n(t) + \frac{\varepsilon^2}{2} \ddot{z}_n(t) + O(\varepsilon^3), \\ \tilde{\tilde{z}}_n(\ell) \equiv z_n(\ell + 2) &\Rightarrow z_n(t) + 2\varepsilon \dot{z}_n(t) + 2\varepsilon^2 \ddot{z}_n(t) + O(\varepsilon^3), \end{aligned} \quad (2.8c)$$

with ε infinitesimal. It is then a matter of standard, if a bit cumbersome, algebra, to verify that the insertion of this *ansatz*, see (2.8b) and (2.8c), in (2.1b) or (2.2b) or (2.3) yields a trivial identity to order $\varepsilon^0 = 1$, while to order ε it reproduces (1.2a) with $\alpha = 1$, reading

$$\ddot{z}_n = -i\omega \dot{z}_n + \sum_{m=1, m \neq n}^N \frac{2(\dot{z}_n + i\omega z_n)(\dot{z}_m + i\omega z_m)}{z_n - z_m}, \quad n = 1, \dots, N. \quad (2.9a)$$

Likewise, the *discrete-time* solution formula (2.5) becomes, in the *continuous-time* limit,

$$\sum_{k=1}^N \frac{\dot{z}_k(0) + i\omega z_k(0)}{z - z_k(0) \exp(-i\omega t)} = \frac{i\omega}{1 - \exp(-i\omega t)} \equiv \frac{i\omega \exp(i\omega t)}{\exp(i\omega t) - 1}, \quad (2.9b)$$

which coincides with (1.2d) with $\alpha = 1$. A terse outline of the derivation of these results is provided at the end of Subsection 3.1. To higher order in ε one would obtain additional relations satisfied by the solution $z_n(t)$ of this *continuous-time* goldfish model, which might alternatively be obtained by differentiating its equations of motion (2.9a).

Remark 2.2. Clearly, for ω real and nonvanishing, this *continuous-time* model, (2.9a), is *isochronous*: see (2.9b) and/or Remark 1.1. This is consistent with the fact that the limiting replacement (2.8b) can be considered to obtain from (2.7) – entailing *isochrony* of the *discrete-time* model – by identifying $\varepsilon\omega$ with $2\pi K/L$ in the context of the replacement (see (2.8)) of the *unit* interval in the *discrete-time* model with the *infinitesimal* time interval ε to make the transition to the *continuous-time* case.

Remark 2.3. At every step of the *discrete-time* evolution the N values of the twice-updated variables $\tilde{z}_n \equiv z_n(\ell+2)$ are given, in terms of the N unupdated variables $z_m \equiv z_m(\ell)$ and the N once-updated variables $\tilde{z}_m \equiv z_m(\ell+1)$, as the N roots of a polynomial, of degree N in its argument z , whose coefficients are explicitly defined in terms of the $2N$ unupdated and once-updated variables: see (2.1) or (2.2) or (2.3) (hereafter – within this important Remark 2.3 – we generally identify, for simplicity, this model only via the version (2.1) of its equations of motion). Hence at every step of this *discrete-time* evolution the *unordered set* of N twice-updated variables \tilde{z}_n is *uniquely* determined, but *not* the value of *each* of them. This implies a *qualitative* difference among the *continuous-time* respectively the *discrete-time* evolutions described by the equations of motions (2.9a) (or, more generally, (1.2a) and (1.3)) respectively by (2.1): in contrast to the *continuous-time* case, the *discrete-time* evolution (2.1) is *only* deterministic in terms of the *unordered set* of N coordinates $z_m(\ell)$, but not for *each* individual coordinate $z_n(\ell)$. Indeed the *continuous-time* Newtonian equations of motion, see for instance (1.2a), determine *uniquely* the value of the acceleration $\ddot{z}_n(t)$ of the n -th moving point in terms of the N positions $z_m(t)$ and the N speeds $\dot{z}_m(t)$ of all moving points; and correspondingly, while the solution formula (1.2d) determines only the *unordered set* of N values $z_n(t)$ as the N roots of a polynomial of degree N , the value of *each* individual coordinate $z_n(t)$ gets then *uniquely* determined by *continuity* in the time variable t . This latter mechanism to identify *uniquely* the value of the coordinate of *each* moving point is instead missing for the *discrete-time* evolution (2.1). On the other hand it is clear that there are appropriate ranges of values of the parameter a and of the $2N$ initial data $z_m(0)$, $\tilde{z}_m(0) \equiv z_m(1)$ – with a sufficiently close to *unity*, the N initial coordinates $z_m(0)$ all sufficiently well separated among themselves, and each $\tilde{z}_m(0) \equiv z_m(1)$ sufficiently close to the corresponding $z_m(0)$, see (2.8) – which cause the evolution yielded by the *discrete-time* goldfish model (2.1) to *mimic closely* that yielded by the *continuous-time* goldfish model (2.9), provided at every step of the *discrete-time* evolution the appropriate identification is made of the value of each twice-updated coordinate $\tilde{z}_n \equiv z_n(\ell+2)$ (among the *unordered set* of N values yielded by the *discrete-time* equations of motion) by an argument of *contiguity* with $\tilde{z}_n \equiv z_n(\ell+1)$ and $z_n \equiv z_n(\ell)$; and likewise an appropriate identification is made by *contiguity* of each coordinate $z_n(\ell+1)$ with the corresponding coordinate $z_n(\ell)$ (among the *unordered set* of N values yielded by Proposition 2.1) – these arguments of *contiguity* taking the place of the *continuity* of $z_n(t)$ as function of t applicable in the *continuous-time* case. But the *contiguity* argument breaks down if the positions at time ℓ of two different points, $z_n(\ell)$ and $z_m(\ell)$ with $n \neq m$, get too close to each other, corresponding to a quasi-collision, or even coincide, corresponding to an actual collision; which is however not featured by the *generic* solution of the *discrete-time* model (2.1) – nor of the standard goldfish models (1.2a) or (1.3) – clearly emerging only for a set of initial conditions $z_n(0)$, $\tilde{z}(0) \equiv z_n(1)$ having *unit* codimension in the $2N$ -dimensional (complex) phase space $\underline{z}, \underline{\tilde{z}}$.

This *important* remark is applicable to all the *discrete-time* models considered below, although it will not be repeated.

Remark 2.4. Several of the formulas written above (in this section) simplify somewhat via the following replacement of the dependent variables:

$$z_n(\ell) \Rightarrow a^\ell z_n(\ell), \quad n = 1, \dots, N.$$

In particular the 3 equivalent versions (2.1b), (2.2b) and (2.3) of the discrete time equations of motion are thereby reformulated to read

$$\begin{aligned} \prod_{j=1}^N \left(\frac{\tilde{z}_n - z_j}{\tilde{z}_n - \tilde{z}_j} \right) &= 1 + a, \quad n = 1, \dots, N, \\ \prod_{j=1}^N \left(\frac{\tilde{z}_j - z_n}{\tilde{z}_j - z_n} \right) &= \frac{1 + a}{a}, \quad n = 1, \dots, N, \\ \prod_{j=1}^N \left(\frac{\tilde{z}_j - \tilde{z}_n}{z_j - \tilde{z}_n} \right) &= -\frac{1}{a}, \quad n = 1, \dots, N, \end{aligned}$$

and correspondingly the formula (2.5) providing the solution of the initial-value problem reads

$$\prod_{k=1}^N \left[\frac{z_n(\ell) - a^{-1}z_k(1)}{z_n(\ell) - z_k(0)} \right] = \frac{a^{\ell-1} - 1}{a^\ell - 1}, \quad n = 1, \dots, N.$$

Somewhat analogous remarks are applicable to all the *discrete-time* models considered below; their explicit implementation is left to the interested reader.

2.2 Second model

In this subsection we treat rather tersely a *discrete-time* dynamical system that generalizes the *discrete-time* goldfish model described in the preceding Subsection 2.1. This generalization amounts to the presence of an additional free parameter, b : indeed, for $b = 0$ one reobtains the model treated in the preceding Subsection 2.1 (hence in this subsection we assume that b does not vanish, $b \neq 0$).

The three equivalent versions of the equations of motion of this model read as follows. The first version identifies the twice updated coordinates $z_n(\ell + 2)$ as the N solution of the following equation in z (amounting to the identification of the N roots of a polynomial of degree N in this variable):

$$\sum_{k=1}^N \left\{ \left(\frac{\tilde{z}_k - az_k}{z - a\tilde{z}_k} \right) \left(\frac{1 + b\tilde{z}_k}{1 + bz_k} \right) \prod_{j=1, j \neq k}^N \left[\left(\frac{\tilde{z}_k - az_j}{\tilde{z}_k - \tilde{z}_j} \right) \left(\frac{1 + b\tilde{z}_j/a}{1 + bz_j} \right) \right] \right\} = 1. \quad (2.10a)$$

The second and third versions consist of the following two systems:

$$\sum_{k=1}^N \left[\left(\frac{\tilde{z}_k - a\tilde{z}_k}{\tilde{z}_k - az_n} \right) \left(\frac{1 + bz_n}{1 + b\tilde{z}_k} \right) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{z}_j - a\tilde{z}_k}{\tilde{z}_j - \tilde{z}_k} \right) \right] = a^{N-1}, \quad n = 1, \dots, N; \quad (2.10b)$$

$$\left[\prod_{j=1}^N \left(\frac{\tilde{z}_j - a\tilde{z}_n}{az_j - \tilde{z}_n} \right) \right] \left[\prod_{j=1, j \neq n}^N \left(\frac{1 + bz_j}{1 + b\tilde{z}_j/a} \right) \right] = a^{N-1} \frac{(1 + b\tilde{z}_n)}{(1 + bz_n)}, \quad n = 1, \dots, N. \quad (2.10c)$$

Remark 2.5. This model – as the original goldfish model (1.2a), and as the model treated above, see Remark 2.1 – is *invariant* under a rescaling of the dependent variables, $z_n \Rightarrow cz_n$ with c an *arbitrary constant*; but only provided the parameter b is also rescaled, $b \Rightarrow b/c$.

The solution of this model is provided by an analog of (the first part of) Proposition 2.1, reading as follows:

Proposition 2.2. *The N values $z_n(\ell)$ of the dependent variables at the discrete time ℓ are the N eigenvalues of the $N \times N$ matrix*

$$U(\ell) = U(0)[aI + bV(0)]^\ell + V(0)[(a-1)I + bV(0)]^{-1} \{[aI + bV(0)]^\ell - I\}, \quad (2.11a)$$

where again (see (2.4))

$$U(0) = \text{diag}[z_n(0)], \quad U_{nm}(0) = \delta_{nm}z_n(0), \quad (2.11b)$$

while the $N \times N$ matrix $V(0)$ is now defined componentwise as follows:

$$[V(0)]_{nm} = \frac{v_m(0)}{1 + bz_m(0)}, \quad n, m = 1, \dots, N, \quad (2.11c)$$

with the quantities $v_m(0)$ defined again as in Subsection 2.1 (see (2.4b) and the sentence following this formula). Let us recall that I is the $N \times N$ unit matrix (whose presence in (2.11a), however, might well be considered pleonastic).

As evidenced by a comparison of (2.11a) with (2.4a), the behavior of the solutions of this model (with $b \neq 0$) are less simple than those of the model discussed in the preceding Subsection 2.1. In particular a *confined* behavior emerges only, see (2.11a), from initial data $z_n(0)$, $\tilde{z}_n(0) \equiv z_n(1)$ implying, via (2.11c) and (2.4b), that *all* the N eigenvalues of the $N \times N$ matrix $aI + bV(0)$ have modulus not larger than *unity*, reading $\exp(-q_n + 2\pi i r_n)$ with the numbers q_n and r_n *real* and the N numbers q_n *nonnegative*, $q_n \geq 0$. If moreover the N numbers q_n *all vanish* and the N numbers r_n are *all rational*, the behavior is *periodic* (but not *isochronous*, since these numbers, q_n and r_n , generally depend on the initial data; see (2.11c) and (2.4b)). While, if some of (but not all) the N numbers q_n are *positive*, and none is *negative*, then the phenomenology we just described (corresponding to the q_n 's *all vanishing*) emerges only *asymptotically*, as $\ell \rightarrow \infty$, up to corrections of order $\exp(-q\ell)$ with q the smallest of the nonvanishing numbers q_n – provided all those r_n 's are *rational* whose corresponding q_n *vanish*.

Let us finally mention that, also for this second model, a transition from the *discrete-time* independent variable ℓ to the *continuous-time* variable t can be performed (as tersely outlined at the end of Subsection 3.2); but the *continuous-time* goldfish-type model obtained in this manner turned out to be, to the best of our knowledge, *new*, hence it seemed appropriate to devote a separate paper to it, see [8].

2.3 Third model

The third model is another one-parameter extension of the model treated in Subsection 2.1 (different from that treated in the preceding Subsection 2.2). Again its *discrete-time* equations of motion can be presented in three equivalent versions.

The first is characterized by this prescription: the twice-updated N coordinates $\tilde{\tilde{z}}_n \equiv z_n(\ell+2)$ are the N roots of the following equation in the variable z ,

$$\sum_{k=1}^N \left[\left(\frac{\tilde{z}_k - a_+ z_k}{z - a_+ \tilde{z}_k} \right) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{z}_k - a_+ z_j}{\tilde{z}_k - \tilde{z}_j} \right) \right] = \frac{1}{a_-}, \quad (2.12a)$$

amounting again to the determination of the N roots of a polynomial of degree N in the variable z . Here and below a_+ and a_- are 2 arbitrary constants.

Remark 2.6. As entailed by a comparison of these *discrete-time* second-order equations of motion with those of the first model, see (2.1a), this third model coincides, for $a_- = 1$, with the first model with $a = a_+$.

A neater formulation of these equations of motion reads (after multiplication by a_+ , via (A.10) with $\zeta_k = a_+^2 z_k$, $\eta_k = a_+ \tilde{z}_k$) as follows:

$$\prod_{j=1}^N \left(\frac{z - a_+^2 z_j}{z - a_+ \tilde{z}_j} \right) = \frac{a_- + a_+}{a_-}. \quad (2.12b)$$

An *equivalent*, second formulation of this model is provided by the following system of N polynomial equations for the twice-updated coordinates $\tilde{\tilde{z}}_n \equiv z_n(\ell + 2)$:

$$\sum_{k=1}^N \left[\left(\frac{\tilde{\tilde{z}}_k - a_+ \tilde{z}_k}{\tilde{\tilde{z}}_k - a_+ z_n} \right) \prod_{j=1, j \neq k}^N \left(\frac{a_+ \tilde{z}_k - \tilde{\tilde{z}}_j}{\tilde{\tilde{z}}_k - \tilde{z}_j} \right) \right] = a_- a_+^{N-1}, \quad n = 1, \dots, N; \quad (2.13a)$$

and a neater version of these equations of motion reads (again via (A.10), but now with $\zeta_k = \tilde{\tilde{z}}_k$, $\eta_k = a_+ \tilde{z}_k$ and z replaced by $a_+^2 z_n$)

$$\prod_{j=1}^N \left(\frac{\tilde{\tilde{z}}_j - a_+^2 z_n}{\tilde{\tilde{z}}_j - a_+ z_n} \right) = (a_+ + a_-) a_+^{N-1}, \quad n = 1, \dots, N. \quad (2.13b)$$

And a third, also *equivalent*, version of these equations of motion reads as follows:

$$\prod_{j=1}^N \left(\frac{\tilde{\tilde{z}}_j - a_+ \tilde{z}_n}{a_+ z_j - \tilde{\tilde{z}}_n} \right) = -a_- a_+^{N-1}, \quad n = 1, \dots, N. \quad (2.14)$$

Remark 2.7. Remark 2.1 also holds for this model.

The solution of the initial-value problem for this *discrete-time* dynamical system is provided by the following

Proposition 2.3. *The N coordinates $z_n(\ell)$ are the N eigenvalues of the $N \times N$ matrix*

$$U(\ell) = (a_+)^{\ell} C_+ + (a_-)^{\ell} C_-, \quad (2.15a)$$

where the two constant (i.e., ℓ -independent) $N \times N$ matrices C_+ and C_- are defined in terms of the $2N$ initial data $z_n(0)$ and $\tilde{z}(0) \equiv z_n(1)$ by the formula

$$C_{\pm} = \pm(a_+ - a_-)^{-1} [U(1) - a_{\mp} U(0)] \quad (2.15b)$$

with the two matrices $U(0)$ and $U(1)$ defined componentwise as follows:

$$[U(0)]_{nm} = \delta_{nm} z_n(0), \quad (2.15c)$$

$$\begin{aligned} [U(1)]_{nm} &= a_+^{1-N} \left[\sum_{k=1}^N z_k(1) - a_+ \sum_{k=1, k \neq n}^N z_k(0) \right] \\ &\quad \times \prod_{j=1, j \neq m}^N \left[\frac{a_+ z_m(0) - z_j(1)}{z_m(0) - z_j(0)} \right], \quad n, m = 1, \dots, N. \end{aligned} \quad (2.15d)$$

Note that we are, for simplicity, assuming that the two coupling constants a_{\pm} are different, $a_+ \neq a_-$ (see (2.15b)).

It is plain from these formulas that, if the two “coupling constants” a_{\pm} (are different and) are conveniently written as follows,

$$a_{\pm} = \exp(-q_{\pm} + 2\pi i r_{\pm}) \quad (2.16)$$

with q_{\pm} and r_{\pm} *real*, then, if the two numbers q_{\pm} are both *nonnegative*, $q_{\pm} \geq 0$, the time evolution of the $N \times N$ matrix $U(\ell)$ is bounded for all values of the *discrete-time* independent variable $\ell = 0, 1, 2, \dots$, hence its N eigenvalues $z_n(\ell)$ are all as well bounded (the motion is confined); if in particular the two numbers q_{\pm} both vanish, $q_{\pm} = 0$, and the two numbers r_{\pm} are both *rational* numbers, $r_{\pm} = K_{\pm}/L_{\pm}$ with K_{+} , L_{+} and K_{-} , L_{-} *coprime integers* (and, for definiteness, $L_{\pm} > 0$), then the *discrete-time* evolution of the matrix $U(\ell)$ is *periodic* (with a period L independent of the initial data, being the minimum common multiple of L_{+} and L_{-} , $L = \text{mcm}[L_{+}, L_{-}]$) hence the *discrete-time* dynamical system (2.12) is *isochronous*; while if, of the two numbers q_{\pm} , one vanishes and the other is positive, $q_{+} = 0$, $q_{-} = q > 0$ respectively $q_{-} = 0$, $q_{+} = q > 0$, and r_{+} respectively r_{-} are *rational* numbers, then the *isochronous* behavior (with period L_{+} respectively L_{-}) only emerges *asymptotically*, as $\ell \rightarrow \infty$, up to corrections of order $\exp(-q\ell)$. While clearly if q_{+} and q_{-} are both *positive* entailing (see (2.16)) $|a_{\pm}| < 0$, then (see (2.15a)) the matrix $U(\ell)$, hence as well all its eigenvalues $z_n(t)$, *vanish* asymptotically (as $t \rightarrow \infty$): $z_n(\infty) = 0$, $n = 1, \dots, N$.

Finally let us mention the transition from this *discrete-time* model to its *continuous-time* counterpart. The treatment is completely analogous to that detailed at the end of Subsection 2.1; except that now (2.8b) must be replaced by

$$a_{\pm} \implies 1 - i\omega_{\pm}\varepsilon$$

with

$$\omega_{+} = \alpha\omega, \quad \omega_{-} = (\alpha - 1)\omega.$$

It is then easily seen that again, at order $\varepsilon^0 = 1$, one gets from (2.12b) or (2.13b) or (2.14) a trivial identity, while at order ε one gets the *continuous-time* goldfish equations of motion (1.2a); and the solution of this model, see Proposition 2.3, reproduces in this *continuous-time* limit the prescription (1.2d).

2.4 Fourth model

The fourth model is also characterized by three equivalent versions of its *discrete-time* equations of motion. The first consists of the following prescription: the twice-updated N coordinates $\tilde{z}_n \equiv z_n(\ell + 2)$ are the N roots of the following equation in the variable z ,

$$\sum_{k=1}^N \left[\frac{\hat{g}_k(z, \tilde{z})}{z - a\tilde{z}_k - b} \right] = \frac{1}{\gamma}, \quad (2.17a)$$

where the N quantities $\hat{g}_k(z, \tilde{z})$ are defined as follows:

$$\begin{aligned} \hat{g}_n(z, \tilde{z}) = & a^{1-N} (\eta \tilde{z}_n + \beta) \left[\prod_{j=1}^N \left(\frac{\tilde{z}_n - a\tilde{z}_j - b}{\eta \tilde{z}_j + \beta} \right) \right] \\ & \times \left[\prod_{j=1, j \neq n}^N \left(\frac{\eta \tilde{z}_j + a\beta - b\eta}{\tilde{z}_n - \tilde{z}_j} \right) \right], \quad n = 1, \dots, N. \end{aligned} \quad (2.17b)$$

Throughout this subsection, the following Subsection 3.4, and Appendix B,

$$a = \alpha + \frac{\eta\rho}{1-\gamma}, \quad b = \frac{\beta\rho}{1-\gamma}, \quad (2.17c)$$

entailing

$$\alpha\beta = a\beta - b\eta, \quad \eta\rho = (a - \alpha)(1 - \gamma), \quad \beta\rho = b(1 - \gamma), \quad (2.17d)$$

where α , β , γ , η and ρ are 5 arbitrary constants (but the 3 constants β , η , ρ only enter as $\beta\rho$ and $\eta\rho$, hence any one of these three constants could be replaced by unity without significant loss of generality); in the following we use interchangeably these constants in order to simplify some formulas.

An *equivalent* formulation of these *discrete-time* equations of motion reads as follows:

$$\sum_{k=1}^N \left[\frac{\check{g}_k(\tilde{z}, \tilde{z})}{(\eta\tilde{z}_k + \beta)(\tilde{z}_k - az_n - b)} \right] = \frac{1}{(\eta z_n + \beta)}, \quad n = 1, \dots, N, \quad (2.18a)$$

with

$$\check{g}_n(\tilde{z}, \tilde{z}) = \frac{a^{1-N}}{\gamma} (\tilde{z}_n - a\tilde{z}_n - b) \prod_{j=1, j \neq n}^N \left(\frac{\tilde{z}_j - a\tilde{z}_n - b}{\tilde{z}_j - \tilde{z}_n} \right), \quad n = 1, \dots, N. \quad (2.18b)$$

It is a matter of trivial algebra to rewrite these equations of motion, (2.18), as follows:

$$\sum_{k=1}^N \left[\frac{1}{(\eta\tilde{z}_k + \beta)(\tilde{z}_k - az_n - b)} \frac{\prod_{j=1}^N (\tilde{z}_j - a\tilde{z}_k - b)}{\prod_{j=1, j \neq k}^N (\tilde{z}_j - \tilde{z}_k)} \right] = \frac{\gamma a^{N-1}}{\eta z_n + \beta}, \quad n = 1, \dots, N. \quad (2.19a)$$

And, as shown at the end of Appendix B, a neater version of this system of equations of motion then reads as follows:

$$\prod_{j=1}^N \left[\frac{\tilde{z}_j - a^2 z_n - b(1+a)}{\tilde{z}_j - az_n - b} \right] - \prod_{j=1}^N \left(\frac{\eta\tilde{z}_j + \alpha\beta}{\eta\tilde{z}_k + \beta} \right) = \gamma a^{N-1} \frac{\eta a z_n + \beta + \eta b}{\eta z_n + \beta}, \quad n = 1, \dots, N. \quad (2.19b)$$

And a third, *equivalent* formulation of these equations of motion reads as follows:

$$\check{g}_n(\tilde{z}, \tilde{z}) = \hat{g}_n(\tilde{z}, \tilde{z}), \quad n = 1, \dots, N, \quad (2.20)$$

with $\check{g}_n(\tilde{z}, \tilde{z})$ respectively $\hat{g}_n(\tilde{z}, \tilde{z})$ defined by (2.18b) respectively (2.17b).

Remark 2.8. Above and below we assume for simplicity that the parameters characterizing this model have *generic* values, for instance $\gamma \neq 0$ and $\gamma \neq 1$ (see (2.17a) and (2.17c)) and $a \neq 0$ (see (2.18b)).

Remark 2.9. This model – as the original goldfish model (1.2a), and as the models treated above, see Remarks 2.1, 2.5 and 2.6 – is *invariant* under a rescaling of the dependent variables, $z_n \Rightarrow cz_n$ with c an *arbitrary constant*; but only provided the parameter β – hence as well the parameter b , see (2.17c) – is also rescaled, $\beta \Rightarrow c\beta$, $b \Rightarrow cb$.

The solution of the initial-value problem for this *discrete-time* dynamical system is provided by the following

Proposition 2.4. *The N coordinates $z_n(\ell)$ are the N eigenvalues of the $N \times N$ matrix*

$$U(\ell) = U(0)P(0, \ell - 1) + \sum_{k=1}^{\ell} [(B\gamma^{k-1} + b)P(k, \ell - 1)], \quad (2.21a)$$

where the $N \times N$ matrix $P(\ell_1, \ell_2)$ is defined as follows

$$P(\ell_1, \ell_2) = \prod_{j=\ell_1}^{\ell_2} (A\gamma^j + a), \quad (2.21b)$$

and the two ℓ -independent $N \times N$ matrices A and B are defined as follows

$$A = \eta V(0) - \frac{\eta}{\beta} b, \quad B = \beta V(0) - b. \quad (2.21c)$$

Here and throughout we use the convention that (for arbitrary finite X_j) $\prod_{j=\ell_1}^{\ell_2} X_j = I$ if $\ell_1 > \ell_2$

and $\sum_{k=k_1}^{k_2} X_j = 0$ if $k_1 > k_2$. As for the two $N \times N$ matrices $U(0)$ and $V(0)$, they are defined in terms of the $2N$ initial data $z_n(0)$ and $\tilde{z}_n(0) \equiv z_n(1)$ as follows:

$$\begin{aligned} U(0) &= Z(0) = \text{diag}[z_n(0)], \\ V(0) &= [\eta Z(0) + \beta]^{-1} \{M(0)Z(1)[M(0)]^{-1} - \alpha Z(0)\}, \end{aligned}$$

with the $N \times N$ matrices $Z(\ell)$ and $M(0)$ defined, componentwise, as follows:

$$\begin{aligned} Z(\ell) &= \text{diag}[z_n(\ell)]; \quad Z_{nm}(\ell) = \delta_{nm} z_n(\ell), \quad n, m = 1, \dots, N, \\ M_{nm}(0) &= \frac{\hat{g}_m(0)}{z_m(1) - a z_n(0) - b}, \quad n, m = 1, \dots, N, \end{aligned}$$

where the notation $\hat{g}_m(0)$ is an abbreviation for $\hat{g}_m(\underline{z}, \underline{\tilde{z}})$, see (2.17b), evaluated at $\underline{z} = \underline{z}(0)$, $\underline{\tilde{z}} = \underline{\tilde{z}}(0) \equiv \underline{z}(1)$. Note that Lemma A.5 (with $f_n = 1$, $g_m = \hat{g}_m(0)$, $\xi_m = z_m(1)$, $\eta_n = a z_n(0) + b$) entails the following componentwise definition of the inverse matrix $[M(0)]^{-1}$:

$$\begin{aligned} \{[M(0)]^{-1}\}_{nm} &= a^{1-N} \left[\frac{z_n(1) - a z_m(0) - b}{\hat{g}_n(0)} \right] \left\{ \prod_{j=1, j \neq n}^N \left[\frac{z_j(1) - a z_m(0) - b}{z_j(1) - z_n(1)} \right] \right\} \\ &\quad \times \left\{ \prod_{j=1, j \neq m}^N \left[\frac{z_n(1) - a z_j(0) - b}{z_m(0) - z_j(0)} \right] \right\}, \quad n, m = 1, \dots, N, \end{aligned}$$

hence an explicit expression of the $N \times N$ matrix $V(0)$ reads, componentwise, as follows:

$$\begin{aligned} V_{nm}(0) &= -\frac{\alpha z_n(0)}{\eta z_n(0) + \beta} \delta_{nm} + \frac{a^{1-N}}{\eta z_n(0) + \beta} \sum_{k=1}^N \left[z_k(1) \left\{ \prod_{j=1, j \neq k}^N \left[\frac{z_j(1) - a z_m(0) - b}{z_j(1) - z_k(1)} \right] \right\} \right. \\ &\quad \times \left. \left\{ \frac{\prod_{j=1, j \neq n}^N [z_k(1) - a z_j(0) - b]}{\prod_{j=1, j \neq m} [z_m(0) - z_j(0)]} \right\} \right]. \end{aligned}$$

Remark 2.10. It is relevant to this expression, (2.21), of the $N \times N$ matrix $U(\ell)$ – whose N eigenvalues provide the N coordinates $z_n(\ell)$ – that (2.21b) and (2.21c) entail

$$P(\ell_1, \ell_2) = Q \operatorname{diag}[p_n(\ell_1, \ell_2)]Q^{-1}, \quad A = Q \operatorname{diag}[a_n]Q^{-1}, \quad B = Q \operatorname{diag}[b_n]Q^{-1},$$

$$p_n(\ell_1, \ell_2) = \prod_{j=\ell_1}^{\ell_2} (a_n \gamma^j + a), \quad a_n = \eta v_n - \frac{\eta}{\beta} b, \quad b_n = \beta v_n - b, \quad n = 1, \dots, N,$$

with v_n the N (ℓ -independent) eigenvalues of the $N \times N$ matrix $V(0)$ and Q the corresponding (ℓ -independent) diagonalizing matrix,

$$V(0) = Q \operatorname{diag}(v_n)Q^{-1}.$$

And let us mention that, also for this fourth model, a transition from the *discrete-time* independent variable ℓ to the *continuous-time* variable t can be performed (see the end of Subsection 3.4). And, as in the case of the second model, also in this case the *continuous-time* goldfish model thereby obtained turned out to be, to the best of our knowledge, *new*. Hence it seemed appropriate to devote to this model a separate paper [9].

3 Proofs

In this section we prove the findings reported in the preceding Section 2.

The basic strategy to obtain all these results goes as follows. The starting point is a *solvable* system of two *matrix* first-order *discrete-time* ODEs, say

$$\tilde{U} = F_1(U, V), \quad \tilde{V} = F_2(U, V), \quad (3.1)$$

where $\ell = 0, 1, 2, \dots$ is the *discrete-time* independent variable, the two dependent variables $U \equiv U(\ell)$, $V \equiv V(\ell)$ are $N \times N$ matrices and of course superimposed tildes denote the updating of the *discrete-time*, $\tilde{U} \equiv U(\ell + 1)$, $\tilde{V} \equiv V(\ell + 1)$. The *solvable* character of this matrix system entails the possibility to obtain *explicitly* the solution of its initial-value problem. Four cases when this is possible – corresponding to 4 simple assignments of the functions $F_1(U, V)$ and $F_2(U, V)$ – are treated in the following 4 subsections. Note that the two functions $F_1(U, V)$, $F_2(U, V)$ are assumed to depend on no other matrix besides U and V (and the unit matrix I); they may of course feature some *scalar* constants, and the order in which the two, generally noncommuting, matrices U and V appear in their definition is of course relevant: see below.

One assumes moreover that the $N \times N$ matrix $U \equiv U(\ell)$ is *diagonalizable* and denotes as $R \equiv R(\ell)$ the diagonalizing $N \times N$ matrix:

$$U \equiv RZR^{-1}, \quad U(\ell) \equiv R(\ell)Z(\ell)[R(\ell)]^{-1}, \quad (3.2a)$$

$$Z = \operatorname{diag}[z_n], \quad Z(\ell) = \operatorname{diag}[z_n(\ell)], \quad (3.2b)$$

where the notation $z_n(\ell)$ for the N eigenvalues of the $N \times N$ matrix $U \equiv U(\ell)$ shall be justified by the identification, see below, of these quantities with the dependent variables of the *discrete-time* dynamical systems introduced above.

Remark 3.1. These formulas entail that the matrix $R(\ell)$ is defined up to right-multiplication by an arbitrary *diagonal* matrix $D(\ell)$, $R(\ell) \Rightarrow R(\ell)D(\ell)$.

Next we introduce the two matrices $M(\ell)$ and $Y(\ell)$ defined as follows:

$$M = R^{-1}\tilde{R}, \quad M(\ell) = [R(\ell)]^{-1}R(\ell + 1), \quad (3.3a)$$

$$V = RY\tilde{R}^{-1}, \quad V(\ell) = R(\ell)Y(\ell)[R(\ell+1)]^{-1}, \quad (3.3b)$$

so that

$$V = RYM^{-1}R^{-1}, \quad V(\ell) = R(\ell)Y(\ell)[M(\ell)]^{-1}[R(\ell)]^{-1}. \quad (3.3c)$$

Remark 3.2. The element of freedom in the definition of the matrix $R(\ell)$, see Remark 3.1, entails that the matrix $M(\ell)$ is defined up to the “gauge transformation” resulting by inserting in its definition (3.3a) the $N \times N$ matrix $[R(\ell)D(\ell)]^{-1} = [D(\ell)]^{-1}[R(\ell)]^{-1}$ in place of the matrix $[R(\ell)]^{-1}$ (and of course $R(\ell+1)D(\ell+1)$ in place of $R(\ell+1)$): hence, as a consequence of the *arbitrary* nature of the *diagonal* matrix $D(\ell)$, out of the N^2 elements of the $N \times N$ matrix $M \equiv M(\ell)$ only $N^2 - N$ are significant. Likewise for the matrix $Y \equiv Y(\ell)$.

One then, by inserting (3.2a) and (3.3c) in (3.1), obtains the following system of two first-order *discrete-time* $N \times N$ matrix evolution equations:

$$M\tilde{Z} = F_1(Z, YM^{-1})M, \quad M\tilde{Y} = F_2(Z, YM^{-1})M\tilde{M}; \quad (3.4)$$

and from these two matrix equations, by making a convenient *ansatz* for the two matrices $M \equiv M(\ell)$ and $Y \equiv Y(\ell)$ in terms of the $2N$ quantities $z_n \equiv z_n(\ell)$ and $\tilde{z}_n \equiv z_n(\ell+1)$ – an *ansatz* which must of course be consistent with these two matrix evolution equations – one obtains a system of N second-order discrete-time evolution equations for the N coordinates $z_n \equiv z_n(\ell)$. This last step is of course only possible for *special* assignments, in the *discrete-time* matrix evolution equations (3.1), of the two matrix functions $F_1(U, V)$ and $F_2(U, V)$, see below.

The *discrete-time* dynamical system thereby obtained is then *solvable*, since the quantities $z_n \equiv z_n(\ell)$ are the N eigenvalues of the $N \times N$ matrix $U \equiv U(\ell)$ which, as solution of the, assumedly *solvable*, matrix evolution system (3.1), can be *explicitly* evaluated. How this works out is shown in detail in the following subsections: in more detail in Subsection 3.1, where the simplest case is treated.

Let us also mention, once and for all, that in the following we will conveniently assume that the matrix U is *initially* diagonal:

$$U(0) = Z(0) \equiv \text{diag}[z_n(0)], \quad (3.5a)$$

implying (up to the ambiguity mentioned above, see Remark 3.1)

$$R(0) = I. \quad (3.5b)$$

Here and throughout I is the $N \times N$ *unit* matrix, i.e., componentwise, $I_{nm} = \delta_{nm}$.

3.1 Solution of the first model

The point of departure to obtain the findings reported in Subsection 2.1 is the following *discrete-time* first-order, linear, matrix system (see (3.1)):

$$\tilde{U} = aU + V, \quad \tilde{V} = V, \quad (3.6a)$$

where a is an arbitrary scalar constant. Note that the second of these two ODEs entails that in this case V is a constant (i.e., ℓ -independent) $N \times N$ matrix, $V(\ell) = V(0)$. It is plain that the solution of the corresponding initial-value problem for the $N \times N$ matrix U reads

$$U(\ell) = U(0)a^\ell + V(0)\frac{a^\ell - 1}{a - 1}. \quad (3.6b)$$

Let us now proceed as indicated in the first part of Section 3. It is then easily seen (via (3.2) and (3.3)) that the first of the two *discrete-time* matrix evolution equations (3.6a) yields (see (3.4)) the matrix equation

$$M\tilde{Z} - aZM = Y, \quad (3.7a)$$

namely, componentwise,

$$M_{nm} = \frac{Y_{nm}}{\tilde{z}_m - az_n}, \quad n, m = 1, \dots, N. \quad (3.7b)$$

Likewise, the second of the two *discrete-time* matrix evolution equations (3.6a) yields the matrix relation

$$Y\tilde{M} = M\tilde{Y}.$$

Via (3.7b) this matrix equation implies the following N^2 relations:

$$\sum_{k=1}^N \left\{ Y_{nk} \tilde{Y}_{km} \left[(\tilde{z}_m - a\tilde{z}_k)^{-1} - (\tilde{z}_k - az_n)^{-1} \right] \right\} = 0, \quad n, m = 1, \dots, N. \quad (3.8)$$

This derivation shows that this system of N^2 *discrete-time* equations of motion is equivalent to the *solvable* equation of motion (3.6a) for the $N \times N$ matrix U ; hence it is just as *solvable*. Note that the dependent variables are now the N coordinates z_n and the N^2 matrix elements Y_{nm} (of which only $N(N-1)$ are significant, see Remark 3.2; so the number of equations and the number of dependent variables tally). To obtain a model that qualifies as *discrete-time* analog of the *continuous-time* goldfish model (2.9a) we need to distill from this system a set of *only* N equations of motion involving *only* the N coordinates z_n . The standard trick to do so (see, for instance, Section 4.2.2 entitled “Goldfishing” of [6]) is to identify – if possible – an *ansatz* which expresses the N^2 components of the matrix Y in terms of the $2N$ quantities z_n , \tilde{z}_n , yielding N equations of motion involving *only* the N coordinates z_n , \tilde{z}_n and $\tilde{\tilde{z}}_n$ – to be interpreted as equations of motion of the *discrete-time* goldfish – and implying that the N^2 equations of motion (3.8) are all satisfied, thanks to these very equations of motion.

An educated guess for such an *ansatz* reads as follows:

$$Y_{nm} = g_m, \quad n, m = 1, \dots, N. \quad (3.9)$$

Note that we reserve at this stage the option to assign the N quantities g_m .

Via this *ansatz* the equations (3.8) become

$$\sum_{k=1}^N \left(\frac{g_k}{\tilde{\tilde{z}}_m - a\tilde{z}_k} - \frac{g_k}{\tilde{z}_k - az_n} \right) = 0, \quad n, m = 1, \dots, N, \quad (3.10a)$$

hence they amount to the following 2 systems, each involving *only* N equations:

$$\sum_{k=1}^N \left(\frac{g_k}{\tilde{z}_k - az_n} \right) = 1, \quad n = 1, \dots, N, \quad (3.10b)$$

$$\sum_{k=1}^N \left(\frac{g_k}{\tilde{\tilde{z}}_n - a\tilde{z}_k} \right) = 1, \quad n = 1, \dots, N. \quad (3.10c)$$

The unit in the right-hand sides could of course be replaced by an arbitrary constant c – of course the same constant in (3.10b) and (3.10c) – but this would merely entail an irrelevant rescaling of g_k by c ; see below.

These are now two sets of N equations, each featuring *linearly* the N quantities g_k , that we like to eliminate in order to obtain a set of N “equations of motion” determining the twice updated coordinates $\tilde{z}_n \equiv z_n(\ell+2)$ in terms of the $2N$ coordinates $z_n \equiv z_n(\ell)$ and $\tilde{z}_n \equiv z_n(\ell+1)$. There are three alternative strategies to achieve this goal. One can solve the first *linear* system thereby obtaining g_k as a function of \underline{z} and $\underline{\tilde{z}}$, and then insert this expression $g_k \equiv \hat{g}_k(\underline{z}, \underline{\tilde{z}})$ in the second system; alternatively, one can solve the second *linear* system, thereby obtaining g_k as a function of $\underline{\tilde{z}}$ and $\underline{\tilde{\tilde{z}}}$, and then insert this expression $g_k \equiv \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$ in the first system; or one can equate the two expressions of g_k obtained solving the first, respectively the second, system, i.e. write $\hat{g}_k(\underline{z}, \underline{\tilde{z}}) = \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$. Clearly the three sets of equations of motion obtained in this manner are *equivalent*, i.e. they characterize the same *discrete-time* evolution of the N coordinates $z_n \equiv z_n(\ell)$; but they may seem quite different (indeed, see (2.1), (2.2) and (2.3)). Note that we introduced a superimposed decoration on the functions $\hat{g}_k(\underline{z}, \underline{\tilde{z}})$ respectively $\check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$ to emphasize that the functional dependence on their arguments is generally different, as implied by their definitions as solutions of (3.10b) respectively of (3.10c).

Let us first of all see what the first approach yields. From (3.10b) one obtains (via Lemma A.1 reported in Appendix A, with $\xi_k = \tilde{z}_k$, $\eta_n = az_n$, $c = 1$) the following expression of $g_k \equiv \hat{g}_k(\underline{z}, \underline{\tilde{z}})$:

$$\hat{g}_k(\underline{z}, \underline{\tilde{z}}) = (\tilde{z}_k - az_k) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{z}_k - az_j}{\tilde{z}_k - \tilde{z}_j} \right), \quad k = 1, \dots, N. \quad (3.11)$$

The insertion of this expression of g_k in (3.10c) yields the equations of motions (2.1a).

Likewise, the second approach yields, from (3.10c) (again via Lemma A.1, but now with $\xi_k = a\tilde{z}_k$, $\eta_n = \tilde{z}_n$, $c = -1$) the following expression of $g_k \equiv \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$:

$$\check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}}) = a^{1-N} (\tilde{z}_k - a\tilde{z}_k) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{\tilde{z}}_j - a\tilde{z}_k}{\tilde{\tilde{z}}_j - \tilde{z}_k} \right), \quad k = 1, \dots, N. \quad (3.12)$$

The second version, (2.2a), of the *discrete-time* equations of motion follows by inserting this expression of g_k in (3.10b).

And the third approach yields, by equating (3.11) to (3.12), the third version, (2.3), of the *discrete-time* equations of motion.

We have seen that the solutions $z_n(\ell)$ of these *discrete-time* equations of motion are provided by the eigenvalues of the $N \times N$ matrix $U(\ell)$, see (3.6b). To prove Proposition 2.1 we must now obtain from (3.6b) (also taking advantage of the *ansatz* (3.9)) the expression (2.4) of this matrix in terms of the initial data $z_n(0)$, $\tilde{z}_n(0) \equiv z_n(1)$ of the *discrete-time* dynamical system.

This requires that we express the two matrices $U(0)$ and $V(0)$ appearing in the right-hand side of (3.6b) in terms of the initial data $z_n(0)$, $\tilde{z}_n(0) \equiv z_n(1)$.

The expression of $U(0)$ is an immediate consequence of (3.5a):

$$[U(0)]_{nm} = \delta_{nm} z_n(0). \quad (3.13)$$

To obtain $V(0)$ we note first of all that (3.3c) and (3.5b) imply

$$V(0) = Y(0)[M(0)]^{-1},$$

while (3.7b) with the *ansatz* (3.9) implies (at $\ell = 0$)

$$M_{nm}(0) = \frac{\hat{g}_m(0)}{z_m(1) - az_n(0)}, \quad n, m = 1, \dots, N,$$

where of course $\hat{g}_m(0)$ stands for $\hat{g}_m(\underline{z}, \underline{\tilde{z}})$, see (3.11), evaluated at $\underline{z} = \underline{z}(0)$, $\underline{\tilde{z}} = \underline{\tilde{z}}(0) \equiv \underline{z}(1)$.

We then evaluate the matrix $[M(0)]^{-1}$ via Lemma A.5 (with $\xi_m = z_m(1)$, $\eta_n = a z_n(0)$, $f_n = 1$ and $g_m = \hat{g}_m(0)$) and, using again the *ansatz* (3.9) (at $\ell = 0$), we obtain the following expression of the $N \times N$ matrix $V(0)$:

$$[V(0)]_{nm} = v_m(0)u_{nm}, \quad n, m = 1, \dots, N,$$

with $v_m(0)$ defined as in Subsection 2.1 (see (2.4b) and the sentence following this formula) and

$$u_{nm} = \sum_{k=1}^N \left\{ \frac{\prod_{j=1, j \neq m}^N [z_k(1) - a z_j(0)]}{\prod_{j=1, j \neq k}^N [z_k(1) - z_j(1)]} \right\}, \quad n, m = 1, \dots, N.$$

But the identity (A.8) (with $\eta_k = z_k(1)$, $\zeta_j = a z_j(0)$) entails $u_{nm} = 1$, hence

$$[V(0)]_{nm} = v_m(0), \quad n, m = 1, \dots, N. \quad (3.14)$$

The insertion of these expressions of $U(0)$ and $V(0)$, (3.13) and (3.14), in (3.6b) yields (2.4), thereby completing the proof of Proposition 2.1.

Let us now provide a terse treatment of the transition from the *discrete-time* equations of motion (2.1b), which we conveniently re-write here as follows,

$$\prod_{j=1}^N \left(\frac{\tilde{z}_n - a^2 z_j}{\tilde{z}_n - a \tilde{z}_j} \right) = 1 + a, \quad n = 1, \dots, N, \quad (3.15)$$

to the *continuous-time* case, see (2.9a). It is then appropriate to treat separately the factor with $j = n$ in the product appearing in the left-hand side of (3.15), and all the other factors with $j \neq n$. The basic equations are (2.8), entailing

$$\begin{aligned} \tilde{z}_n - a^2 z_n &= 2\varepsilon(\dot{z}_n + i\omega z_n) + \varepsilon^2(2\ddot{z}_n + \omega^2 z_n) + O(\varepsilon^3), \\ \tilde{z}_n - a \tilde{z}_n &= \varepsilon(\dot{z}_n + i\omega z_n) + \frac{\varepsilon^2}{2}(3\ddot{z}_n + 2i\omega \dot{z}_n) + O(\varepsilon^3), \\ \tilde{z}_n - a^2 z_j &= z_n - z_j + 2\varepsilon(\dot{z}_n + i\omega z_j) + O(\varepsilon^2), \quad j \neq n, \\ \tilde{z}_n - a \tilde{z}_j &= z_n - z_j + \varepsilon(2\dot{z}_n - \dot{z}_j + i\omega z_j) + O(\varepsilon^2), \quad j \neq n. \end{aligned}$$

Hence, after a little algebra,

$$\frac{\tilde{z}_n - a^2 z_n}{\tilde{z}_n - a \tilde{z}_n} = 2 + \varepsilon \frac{-\ddot{z}_n - 2i\omega \dot{z}_n + \omega^2 z_n}{\dot{z}_n + i\omega z_n} + O(\varepsilon^2), \quad (3.16a)$$

$$\frac{\tilde{z}_n - a^2 z_j}{\tilde{z}_n - a \tilde{z}_j} = 1 + \varepsilon \frac{\dot{z}_j + i\omega z_j}{z_n - z_j} + O(\varepsilon^2), \quad j \neq n, \quad (3.16b)$$

implying

$$\begin{aligned} \prod_{j=1}^N \left(\frac{\tilde{z}_n - a^2 z_j}{\tilde{z}_n - a \tilde{z}_j} \right) &= \left(2 + \varepsilon \frac{-\ddot{z}_n - 2i\omega \dot{z}_n + \omega^2 z_n}{\dot{z}_n + i\omega z_n} \right) \\ &\quad \times \left[1 + \varepsilon \sum_{j=1, j \neq n}^N \left(\frac{\dot{z}_j + i\omega z_j}{z_n - z_j} \right) \right] + O(\varepsilon^2). \end{aligned} \quad (3.16c)$$

While of course

$$1 + a = 2 - i\omega\varepsilon, \quad (3.16d)$$

see (2.8b). It is then clear that the insertion of these two formulas, (3.16c) and (3.16d), in (3.15) yields, at order $\varepsilon^0 = 1$, the trivial identity $2 = 2$, and at order ε the equations of motion of the *continuous-time* goldfish model (2.9a).

In an analogous manner one reobtains (2.9a) from (2.2b) or from (2.3).

Let us also show that (2.5), which we rewrite here conveniently as follows,

$$\prod_{k=1}^N \left[\frac{z - a^\ell z_k(1)a^{-1}}{z - a^\ell z_k(0)} \right] = \frac{a^\ell a^{-1} - 1}{a^\ell - 1}, \quad (3.17)$$

yields, in the *continuous-time* limit, (2.9b). Indeed the relation $a = 1 - i\varepsilon\omega$ (see (2.8b)) entails

$$\begin{aligned} a^{-1} &= 1 + i\varepsilon\omega + O(\varepsilon^2), \\ a^\ell &= \exp(-i\omega t) \left(1 - \varepsilon \frac{\omega^2 t}{t} \right) + O(\varepsilon^2) \end{aligned}$$

(via the first of the three relations (2.8a)), and

$$z_k(1) = z_k(0) + \varepsilon \dot{z}(0) + O(\varepsilon^2)$$

(via the second of the three relations (2.8c), with $\ell = 0$). Via these three relations (3.17) becomes

$$\begin{aligned} \prod_{k=1}^N \left[\frac{z - \exp(-i\omega t) (1 - \varepsilon\omega^2 t/2) \{z_k(0) + \varepsilon[\dot{z}_k(0) + i\omega z_k(0)]\} + O(\varepsilon^2)}{z - \exp(-i\omega t) (1 - \varepsilon\omega^2 t/2) z_k(0) + O(\varepsilon^2)} \right] \\ = \frac{\exp(-i\omega t) (1 - \varepsilon\omega^2 t/2) (1 + i\varepsilon\omega) - 1 + O(\varepsilon^2)}{\exp(-i\omega t) (1 - \varepsilon\omega^2 t/2) - 1 + O(\varepsilon^2)}, \end{aligned}$$

i.e. (dividing each numerator by the corresponding denominator)

$$\prod_{k=1}^N \left[1 - \varepsilon \frac{[\dot{z}_k(0) + i\omega z_k(0)]}{z - \exp(-i\omega t) z_k(0)} + O(\varepsilon^2) \right] = 1 + \varepsilon \frac{i\omega}{\exp(-i\omega t) - 1} + O(\varepsilon^2).$$

Clearly to order $\varepsilon^0 = 1$ this yields the trivial identity $1 = 1$, and to order ε just the formula (2.9b).

Let us end this subsection by pointing out that there is another *ansatz* that allows to transform the system of N^2 equations (3.8) into two separate systems of N equations, but only in the special case $a = 1$. This alternative *ansatz* reads (instead of (3.9))

$$Y_{nm} = \frac{f_m}{\tilde{z}_m - z_n}, \quad n, m = 1, \dots, N,$$

entailing (but only provided $a = 1$) the replacement of the system of N^2 equations (3.8) with the following two systems of N equations:

$$\begin{aligned} \sum_{k=1}^N \left[\frac{f_k}{(\tilde{z}_k - z_n)^2} \right] &= 1, \quad n = 1, \dots, N, \\ \sum_{k=1}^N \left[\frac{f_k}{(\tilde{z}_n - \tilde{z}_k)^2} \right] &= 1, \quad n = 1, \dots, N. \end{aligned}$$

But, as indicated at the end of Section 1, we postpone the treatment of the corresponding class of *discrete-time* dynamical systems to a separate paper.

3.2 Solution of the second model

The proof of the findings reported in Subsection 2.2 is analogous to that provided above, see Subsection 3.1, so our treatment in this subsection is quite terse, being limited to indicate the changes with respect to that reported in the preceding Subsection 3.1. Now the system of matrix evolution equations (3.6a) is generalized to read

$$\tilde{U} = U(aI + bV) + V, \quad \tilde{V} = V; \quad (3.18)$$

hence its solution is given by (2.11a). Clearly this evolution equation, (3.18), respectively its solution, (2.11a), reduce to (3.6a) respectively to (3.6b) when b vanishes.

The rest of the treatment is analogous. (3.7a) is now generalized to read

$$M\tilde{Z} - aZM = (I + bZ)Y,$$

hence it yields, in place of (3.7b),

$$M_{nm} = \left(\frac{1 + bz_n}{\tilde{z}_m - az_n} \right) Y_{nm}, \quad n, m = 1, \dots, N.$$

In place of (3.10a) (again via the *ansatz* (3.9)) one now has

$$\sum_{k=1}^N \left[\frac{g_k(1 + bz_k)}{\tilde{z}_m - a\tilde{z}_k} - \frac{g_k(1 + bz_n)}{\tilde{z}_k - az_n} \right] = 0, \quad n, m = 1, \dots, N, \quad (3.19a)$$

hence in place of (3.10b) and (3.10c) one gets the two sets of N equations

$$\sum_{k=1}^N \left[\frac{g_k}{\tilde{z}_k - az_n} \right] = \frac{1}{1 + bz_n}, \quad n = 1, \dots, N, \quad (3.19b)$$

$$\sum_{k=1}^N \left[\frac{g_k(1 + bz_k)}{\tilde{z}_n - a\tilde{z}_k} \right] = 1, \quad n = 1, \dots, N. \quad (3.19c)$$

By solving the first set one obtains (via Lemma A.2, with $\xi_k = \tilde{z}_k$, $\eta_n = az_n$, $c_n = 1/(1 + bz_n)$, and then the identity (A.7) with $z = -a/b$, $\eta_k = az_k$, $\zeta_j = \tilde{z}_j$) the following expression of $g_k \equiv \hat{g}_k(\underline{z}, \underline{\tilde{z}})$:

$$\hat{g}_k(\underline{z}, \underline{\tilde{z}}) = \left(\frac{\tilde{z}_k - az_k}{1 + bz_k} \right) \prod_{j=1, j \neq k}^N \left[\left(\frac{\tilde{z}_k - az_j}{\tilde{z}_k - \tilde{z}_j} \right) \left(\frac{1 + b\tilde{z}_j/a}{1 + bz_j} \right) \right]. \quad (3.20a)$$

By solving instead the second set one obtains (via Lemma A.1, with $\xi_k = -a\tilde{z}_k$, $\eta_n = -\tilde{z}_n$, $c = 1$ and g_k replaced by $g_k(1 + b\tilde{z}_k)$) the following expression of $g_k \equiv \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{z}})$:

$$\check{g}_k(\underline{\tilde{z}}, \underline{\tilde{z}}) = a^{1-N} \left(\frac{\tilde{z}_k - a\tilde{z}_k}{1 + b\tilde{z}_k} \right) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{z}_j - a\tilde{z}_k}{\tilde{z}_j - \tilde{z}_k} \right). \quad (3.20b)$$

The three versions, (2.10), of the equations of motion reported in Subsection 2.2 then follow by inserting (3.20a) in (3.19c), by inserting (3.20b) in (3.19b), and by equating (3.20a) to (3.20b).

Next, let us prove Proposition 2.2. One proceeds again in close analogy to the treatment of the preceding Subsection 3.1, hence we only mention where the treatment here differs from that provided there. It is easily seen that the expression of $U(0)$ is the same as that given there, see (3.13), while the expression of $V(0)$ (because one must now use Lemma A.5 with

$f_n = 1 + bz_n(0)$ rather than $f_n = 1$) is now given by (2.11c). The insertion of these expressions of $U(0)$ and $V(0)$ in (3.18) reproduce (2.11), thereby proving Proposition 2.2.

Let us end this subsection by outlining what happens in the *continuous-time* limit which obtains by setting

$$a = 1 + \varepsilon b\eta, \quad V(0) = \varepsilon B$$

with ε infinitesimal, and correspondingly replacing the *discrete-time* matrix evolution equation (3.18) with the matrix ODE

$$\dot{U} = bU(\eta I + B) + B, \quad (3.21a)$$

the solution of which reads

$$U(t) = U(0) \exp[b(\eta I + B)t] + B[b(\eta I + B)]^{-1} \{\exp[b(\eta I + B)t] - I\}. \quad (3.21b)$$

As already mentioned in Subsection 2.2, the (*continuous-time*) goldfish-type model obtainable by focussing appropriately on the evolution of the N eigenvalues $z_n(t)$ of this $N \times N$ matrix $U(t)$ (evolving according to (3.21a)) was, to the best of our knowledge, *new*, when the *solvable* matrix evolution equation (3.21a) was identified as *continuous-time* limit of (3.18); its treatment is provided in [8].

3.3 Solution of the third model

The starting point is the following linear system of two *discrete-time* matrix evolution equations:

$$\tilde{U} = a_+ U + \beta V, \quad \tilde{V} = a_- V, \quad (3.22a)$$

where the 3 constants a_\pm, β are *a priori* arbitrary ($\beta \neq 0$). It is easily seen that the solution of the initial-value problem for U reads as follows (with an analogous formula for V):

$$U(\ell) = a_+^\ell C_+ + a_-^\ell C_-, \quad (3.22b)$$

and the two constant matrices C_\pm given by (2.15b).

Remark 3.3. This solution $U(\ell)$ of the initial-value problem for the *discrete-time* $N \times N$ matrix evolution equation (3.22a) depends only on the 2 constants a_\pm : see (3.22b) and (2.15b). Indeed the system of two first-order *discrete-time* evolution equations (3.22a) is easily seen to correspond to the single second-order evolution equation

$$\tilde{\tilde{U}} - (a_+ + a_-)\tilde{U} + a_+a_-U = 0,$$

from which the constant β has disappeared (but note that this *second-order* matrix ODE obtains only if $\beta \neq 0$; indeed if $\beta = 0$, U satisfies a *first-order* evolution equation, see the first of the two equations (3.22a).

Remark 3.4. For $a_+ = a, \beta = 1, a_- = 1$, the system (3.22a) coincides with (3.6a) hence this model reduces to the first model, confirming Remark 2.6.

We then proceed as in the first part of Section 3. It is then easily seen that the two matrix evolution equations (3.22a) become

$$\begin{aligned} M\tilde{Z} &= a_+ ZM + \beta Y, \\ M\tilde{Y} &= a_- Y\tilde{M}. \end{aligned}$$

Via (3.2b) the first of these two equations reads, componentwise,

$$M_{nm} = \frac{\beta Y_{nm}}{\tilde{z}_m - a_+ z_n}, \quad n, m = 1, \dots, N, \quad (3.23)$$

and using this formula it is easily seen that the second can be written, componentwise, as follows:

$$\sum_{k=1}^N \left[Y_{nk} \tilde{Y}_{km} \left(\frac{a_-}{\tilde{z}_m - a_+ \tilde{z}_k} - \frac{1}{\tilde{z}_k - a_+ z_n} \right) \right] = 0, \quad n, m = 1, \dots, N. \quad (3.24)$$

At this point we use again the *ansatz* (3.9) for the matrix Y_{nm} , the consistency of which is vindicated by the subsequent developments. Here the N quantities g_m are again *a priori* arbitrary; they shall be determined as functions of the $2N$ un-updated and once-updated coordinates $z_n \equiv z_n(\ell)$ and $\tilde{z}_n \equiv z_n(\ell + 1)$, or alternatively of the once and twice updated coordinates $\tilde{z}_n \equiv z_n(\ell + 1)$ and $\tilde{\tilde{z}}_n \equiv z_n(\ell + 2)$, see below. It is indeed immediately seen that via the *ansatz* (3.9) the system of N^2 equations (3.24) becomes

$$\sum_{k=1}^N \left(\frac{a_- g_k}{\tilde{z}_m - a_+ \tilde{z}_k} - \frac{g_k}{\tilde{z}_k - a_+ z_n} \right) = 0, \quad n, m = 1, \dots, N; \quad (3.25a)$$

hence it can be replaced by the following two separated systems of *only* N equations:

$$\sum_{k=1}^N \left(\frac{g_k}{\tilde{z}_k - a_+ z_n} \right) = 1, \quad n = 1, \dots, N, \quad (3.25b)$$

$$\sum_{k=1}^N \left(\frac{g_k}{\tilde{\tilde{z}}_n - a_+ \tilde{z}_k} \right) = \frac{1}{a_-}, \quad n = 1, \dots, N. \quad (3.25c)$$

The first, (3.25b), of these two systems defines uniquely the N quantities $g_k \equiv \hat{g}_k(\underline{z}, \underline{\tilde{z}})$, yielding again, via (A.13), the expression (3.11) (with a replaced by a_+):

$$\hat{g}_k(\underline{z}, \underline{\tilde{z}}) = (\tilde{z}_k - a_+ z_k) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{z}_k - a_+ z_j}{\tilde{z}_k - \tilde{z}_j} \right), \quad k = 1, \dots, N. \quad (3.26)$$

Insertion of this expression in the second, (3.25c), of the two systems written just above then yields the evolution equation (2.12a). The identification of the third *discrete-time* dynamical system of goldfish type, see (2.12a), is thereby accomplished.

The second version, (2.13a), of this model obtains by solving for $g_k \equiv \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$ the second, (3.25c), of the two systems written above (via Lemma A.1, with $\xi_k = a_+ \tilde{z}_k$, $\eta_n = \tilde{\tilde{z}}_n$ and $c = -1/a_-$), thereby obtaining

$$\check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}}) = \left(\frac{\tilde{\tilde{z}}_k - a_+ \tilde{z}_k}{a_-} \right) \prod_{j=1, j \neq k}^N \left(\frac{a_+ \tilde{z}_k - \tilde{\tilde{z}}_j}{a_+ \tilde{z}_k - a_+ \tilde{z}_j} \right), \quad k = 1, \dots, N; \quad (3.27)$$

and by then inserting this expression of g_k in (3.25b).

And the third version, (2.14), of this model obtains by equating (3.26) to (3.27).

As for the proof of Proposition 2.3, it follows immediately from the above treatment, see in particular (2.15); there only remains to justify the identification of the two matrices $U(0)$ and $U(1)$, see (2.15c) and (2.15d).

The first of the two formulas, (2.15c), is just (3.5a).

To prove the second, (2.15d), we note that (3.5a) and (3.3a) entail (for $\ell = 0$)

$$R(1) = M(0), \quad (3.28)$$

while (3.2) (with $\ell = 0$) reads

$$U(1) = R(1)Z(1)[R(1)]^{-1}. \quad (3.29a)$$

Hence (via (3.28))

$$U(1) = M(0)Z(1)[M(0)]^{-1}. \quad (3.29b)$$

We now note that (3.23) and the *ansatz* (3.9) imply that the $N \times N$ matrix $M(0)$ is defined componentwise as follows:

$$M_{nm}(0) = \frac{g_m(0)}{z_m(1) - a_+ z_n(0)}, \quad n, m = 1, \dots, N, \quad (3.30a)$$

with (see (3.26))

$$g_m(0) = [z_m(1) - a_+ z_m(0)] \prod_{j=1, j \neq m}^N \left[\frac{z_m(1) - a_+ z_j(0)}{z_m(1) - z_j(1)} \right], \quad m = 1, \dots, N, \quad (3.30b)$$

so that

$$M_{nm}(0) = \frac{z_m(1) - a_+ z_m(0)}{z_m(1) - a_+ z_n(0)} \prod_{j=1, j \neq m}^N \left[\frac{z_m(1) - a_+ z_j(0)}{z_m(1) - z_j(1)} \right], \quad n, m = 1, \dots, N. \quad (3.30c)$$

Next, we note that the expression (3.30a) of the matrix $M(0)$ entails – via Lemma A.5 (with $f_n = 1$, $g_m = g_m(0)$, $\xi_m = z_m(1)$, $\eta_n = a_+ z_n(0)$) and the expression of $g_m(0)$ given above – that its inverse, appearing in the right-hand-side of (3.29b), is explicitly given, componentwise, as follows:

$$\{[M(0)]^{-1}\}_{nm} = a_+^{1-N} \left[\frac{a_+ z_m(0) - z_m(1)}{a_+ z_m(0) - z_n(1)} \right] \prod_{j=1, j \neq m}^N \left[\frac{a_+ z_m(0) - z_j(1)}{z_m(0) - z_j(0)} \right], \quad n, m = 1, \dots, N. \quad (3.30d)$$

The insertion of this formula and (3.30c) in (3.29b) entails that the matrix $U(1)$ reads componentwise as follows:

$$U_{nm}(1) = a_+^{1-N} \prod_{j=1, j \neq m}^N \left[\frac{a_+ z_m(0) - z_j(1)}{z_m(0) - z_j(0)} \right] \times \sum_{k=1}^N \left\{ z_k(1) \frac{\prod_{j=1, j \neq n} [z_k(1) - a_+ z_j(0)]}{\prod_{j=1, j \neq k} [z_k(1) - z_j(1)]} \right\}, \quad n, m = 1, \dots, N.$$

And it is then immediately seen that this formula yields, via the identity (A.12) (with $\eta_k = z_k(1)$, $k = 1, \dots, N$; $\zeta_j = a_+ z_j(0)$, $j = 1, \dots, n-1, n+1, \dots, N$), the formula (2.15d).

3.4 Solution of the fourth model

The treatment in this subsection is rather terse, since it is analogous to that of the preceding subsections; and the notation is of course analogous. But now the starting point is the following nonlinear system of two *discrete-time* matrix evolution equations:

$$\tilde{U} = \alpha U + \beta V + \eta UV, \quad \tilde{V} = \rho + \gamma V, \quad (3.31)$$

featuring the 5 arbitrary constants $\alpha, \beta, \eta, \rho, \gamma$ (which, as noted in Subsection 2.4, can be reduced to 4 by taking advantage of the freedom to rescale V).

It is a standard task to see that the solution of the initial-value problem for this matrix system reads as follows:

$$V(\ell) = \gamma^\ell V(0) + \rho \frac{\gamma^\ell - 1}{\gamma - 1} I,$$

with $U(\ell)$ given by (2.21).

We now proceed again as in the first part of Section 3. Via (3.3a) and (3.3c) we get from (3.31) the two matrix equations

$$M\tilde{Z} = \alpha ZM + (\beta + \eta Z)Y, \quad (3.32a)$$

$$M\tilde{Y} = \rho M\tilde{M} + \gamma Y\tilde{M}. \quad (3.32b)$$

The first, (3.32a), of these two matrix equations entails, componentwise,

$$Y_{nm} = \left(\frac{\tilde{z}_m - \alpha z_n}{\eta z_n + \beta} \right) M_{nm}, \quad n, m = 1, \dots, N;$$

hence the second, (3.32b), of these two matrix equations yields (when written componentwise) the following N^2 equations:

$$\sum_{k=1}^N \left[M_{nk} \tilde{M}_{km} \left(\frac{\tilde{z}_m - \alpha \tilde{z}_k}{\eta \tilde{z}_k + \beta} - \rho - \gamma \frac{\tilde{z}_k - \alpha z_n}{\eta z_n + \beta} \right) \right] = 0, \quad n, m = 1, \dots, N, \quad (3.33a)$$

which, as can be easily verified, can be conveniently rewritten as follows:

$$\sum_{k=1}^N \left[M_{nk} \tilde{M}_{km} \left(\frac{\tilde{z}_m - a \tilde{z}_k - b}{\eta \tilde{z}_k + \beta} - \gamma \frac{\tilde{z}_k - a z_n - b}{\eta z_n + \beta} \right) \right] = 0, \quad n, m = 1, \dots, N, \quad (3.33b)$$

with the two constants a and b defined by (2.17c).

Next, we make the following *ansatz* for the matrix M_{nm} :

$$M_{nm} = \frac{g_m}{\tilde{z}_m - a z_n - b}, \quad n, m = 1, \dots, N. \quad (3.34)$$

Here the N quantities g_m are *a priori* arbitrary; they shall be determined as functions of the $2N$ un-updated and once-updated coordinates $z_n \equiv z_n(\ell)$ and $\tilde{z}_n \equiv z_n(\ell+1)$, or of the once and twice updated coordinates $\tilde{z}_n \equiv z_n(\ell+1)$ and $\tilde{\tilde{z}}_n \equiv z_n(\ell+2)$, see below. It is indeed immediately seen that via this *ansatz* (3.34) the system of N^2 equations (3.33b) can be rewritten as follows:

$$\sum_{k=1}^N \left[\frac{g_k(\eta z_n + \beta)}{(\eta \tilde{z}_k + \beta)(\tilde{z}_k - a z_n - b)} - \gamma \frac{g_k}{(\tilde{\tilde{z}}_m - a \tilde{z}_k - b)} \right] = 0, \quad n, m = 1, \dots, N; \quad (3.35a)$$

hence it can be replaced by the following two separated systems of *only* N equations:

$$\sum_{k=1}^N \left[\frac{g_k}{(\eta \tilde{z}_k + \beta)(\tilde{z}_k - a z_n - b)} \right] = \frac{1}{(\eta z_n + \beta)}, \quad n = 1, \dots, N, \quad (3.35b)$$

$$\sum_{k=1}^N \left(\frac{g_k}{\tilde{z}_n - a \tilde{z}_k - b} \right) = \frac{1}{\gamma}, \quad n = 1, \dots, N. \quad (3.35c)$$

The first of these two systems defines uniquely the N quantities $g_k \equiv \hat{g}_k(\underline{z}, \underline{\tilde{z}})$, yielding their expression (2.17b) (see Appendix B for a proof). The second equation then yields the evolution equation (2.17a).

The alternative possibility is to determine (via Lemma A.1, with $\xi_k = a \tilde{z}_k$, $\eta_n = \tilde{z}_n - b$, $c = -1/\gamma$) the quantities $g_k \equiv \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$ as solutions of the second system, yielding the formula (2.18b); and to then insert these expressions of $g_k \equiv \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$ in the first system of equations. Clearly in this manner one arrives at the equations of motion (2.19a).

And a third possibility is of course to equate (2.17b) to (2.18b), see (2.20).

The identification of the three variants of the equations of motion of the fourth *discrete-time* dynamical system of goldfish type, see Subsection 2.4, is thereby accomplished.

The proof of Proposition 2.4 follows immediately from the above treatment; and we trust that the identification in terms of the $2N$ initial data $z_n(0)$ and $z_n(1)$ of the two matrices $U(0)$ and $V(0)$, see (2.4) and (2.4), is sufficiently obvious (also in the light of the analogous treatment in the preceding subsections of this section) not to require an explicit justification here.

We end this subsection with a terse mention of the *continuous-time* model that obtains from that treated above (in this subsection) via the limiting transition from *discrete* to *continuous* time. The point of departure for the treatment of the *continuous-time* dynamical system of goldfish type obtained in this manner is the following *continuous-time* system of two first-order matrix evolution equations

$$\dot{U} = a_1 U + a_2 V + a_3 UV, \quad \dot{V} = a_4 + a_5 V,$$

that obtains from (3.31) via the assignments $t \Rightarrow \varepsilon \ell$, $U(\ell) \Rightarrow U(t)$, $V(\ell) \Rightarrow V(t)$, $\alpha = 1 + \varepsilon a_1$, $\beta = \varepsilon a_2$, $\eta = \varepsilon a_3$, $\rho = \varepsilon a_4$, $\gamma = 1 + \varepsilon a_5$, with ε infinitesimal. The resulting model of goldfish type was, to the best of our knowledge, *new*; a detailed treatment of it is provided in [9].

4 Outlook

In this paper we have introduced and tersely analyzed 4 different *discrete-time* dynamical systems of goldfish type. The possibility to identify other *discrete-time* evolution equations amenable to exact treatment by variations of the methodology used in this paper is open: let us outline here an avenue to such generalizations.

Consider the system of two $N \times N$ matrix *discrete-time* first-order evolution equations

$$\sum_{s=1}^{S_1} [F_{1,s}(U) F_{2,s}(\tilde{U})] = \sum_{s=1}^{S_2} [\Phi_{1,s}(U) V \Phi_{2,s}(\tilde{U})], \quad (4.1a)$$

$$\sum_{s=1}^{S_3} [F_{3,s}(U) F_{4,s}(\tilde{U})] \tilde{V} = \Phi_3(U) V \Phi_4(\tilde{U}), \quad (4.1b)$$

where the two $N \times N$ matrices $U \equiv U(\ell)$ and $V(\ell)$ are the dependent variables, $\ell = 0, 1, 2, \dots$ is the independent *discrete-time* variable, S_1, S_2, S_3 are 3 arbitrary positive integers, $F_{1,s}(u)$, $F_{2,s}(u)$, $F_{3,s}(u)$, $F_{4,s}(u)$ and $\Phi_{1,s}(u)$, $\Phi_{2,s}(u)$, $\Phi_3(u)$, $\Phi_4(u)$ are $2(S_1 + S_2 + S_3 + 1)$ *a priori*

arbitrary (scalar) functions of their (scalar) argument u (of course becoming $N \times N$ matrices when the scalar u is replaced by an $N \times N$ matrix). Then introduce the eigenvalues $z_n(\ell)$ of the matrix $U(\ell)$, as well as the matrices $Z \equiv Z(\ell)$, $R \equiv R(\ell)$, $M \equiv M(\ell)$ and $Y \equiv Y(\ell)$, as above (see (3.2) and (3.3)). It is then plain that the matrix evolution equation (4.1a) becomes

$$\sum_{s=1}^{S_1} [F_{1,s}(Z) M F_{2,s}(\tilde{Z})] = \sum_{s=1}^{S_2} [\Phi_{1,s}(Z) Y \Phi_{2,s}(\tilde{Z})] \quad (4.2a)$$

entailing componentwise

$$M_{nm} = \frac{\sum_{s=1}^{S_2} [\Phi_{1,s}(z_n) \Phi_{2,s}(\tilde{z}_m)]}{\sum_{s=1}^{S_1} [F_{1,s}(z_n) F_{2,s}(\tilde{z}_m)]} Y_{nm}, \quad n, m = 1, \dots, N. \quad (4.2b)$$

Likewise (4.1b) becomes

$$\sum_{s=1}^{S_3} [F_{3,s}(Z) M F_{4,s}(\tilde{Z})] \tilde{Y} = \Phi_3(Z) Y \Phi_4(\tilde{Z}) \tilde{M},$$

entailing componentwise (via (4.2b))

$$\begin{aligned} \sum_{k=1}^N \left\{ Y_{nk} \tilde{Y}_{km} \left[\sum_{s=1}^{S_2} F_{3,s}(z_n) F_{4,s}(\tilde{z}_k) \right] \frac{\sum_{\sigma=1}^{S_2} [\Phi_{1,\sigma}(z_n) \Phi_{2,\sigma}(\tilde{z}_k)]}{\sum_{\sigma=1}^{S_1} [F_{1,\sigma}(z_n) F_{2,\sigma}(\tilde{z}_k)]} \right\} \\ = \Phi_3(z_n) \sum_{k=1}^N \left\{ Y_{nk} \tilde{Y}_{km} \Phi_4(\tilde{z}_k) \frac{\sum_{s=1}^{S_2} [\Phi_{1,s}(\tilde{z}_k) \Phi_{2,s}(\tilde{z}_m)]}{\sum_{s=1}^{S_1} [F_{1,s}(\tilde{z}_k) F_{2,s}(\tilde{z}_m)]} \right\}, \quad n, m = 1, \dots, N. \end{aligned}$$

And via the *ansatz* $Y_{nm} = g_m$ (see (3.9)) this system of N^2 equations can clearly be replaced by the following two systems of N linear algebraic equations for the N quantities g_k :

$$\sum_{k=1}^N \left\{ g_k \left[\sum_{s=1}^{S_2} F_{3,s}(z_n) F_{4,s}(\tilde{z}_k) \right] \frac{\sum_{\sigma=1}^{S_2} [\Phi_{1,\sigma}(z_n) \Phi_{2,\sigma}(\tilde{z}_k)]}{\sum_{\sigma=1}^{S_1} [F_{1,\sigma}(z_n) F_{2,\sigma}(\tilde{z}_k)]} \right\} = \Phi_3(z_n), \quad n = 1, \dots, N, \quad (4.3a)$$

$$\sum_{k=1}^N \left\{ g_k \Phi_4(\tilde{z}_k) \frac{\sum_{s=1}^{S_2} [\Phi_{1,s}(\tilde{z}_k) \Phi_{2,s}(\tilde{z}_n)]}{\sum_{s=1}^{S_1} [F_{1,s}(\tilde{z}_k) F_{2,s}(\tilde{z}_n)]} \right\} = 1, \quad n = 1, \dots, N. \quad (4.3b)$$

One can then solve the first, (4.3a), respectively the second, (4.3b), of these two *linear* systems for the N quantities g_k , getting the expressions $g_k = \hat{g}_k(\underline{z}, \underline{\tilde{z}})$ respectively $g_k = \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$. One arrives thereby at three *equivalent* systems of N *discrete-time* second-order evolution equations for the N coordinates $z_n(\ell)$: (i) by inserting the expression $\hat{g}_k(\underline{z}, \underline{\tilde{z}})$ in (4.3b); (ii) by inserting the expression $\check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$ in (4.3a); (iii) by setting $\hat{g}_n(\underline{z}, \underline{\tilde{z}}) = \check{g}_n(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$. While of course the evolution in the discrete time ℓ entailed by these equations of motions corresponds to the evolution of

the eigenvalues of the matrix $U(\ell)$ solution of the matrix evolution equation (4.1a) (with the *ansatz* (3.9) properly taken into account). Hence if that matrix evolution equation, (4.1a), is *solvable*, the *discrete-time* dynamical system described by these three equivalent sets of second-order evolution equations is as well *solvable*.

One has thereby identified a *discrete-time* solvable dynamical system. A remaining open question is the extent to which its equations of motion can be exhibited in reasonably neat form: this depends on the extent that the two quantities $g_k \equiv \hat{g}_k(\underline{z}, \underline{\tilde{z}})$ respectively $g_k \equiv \check{g}_k(\underline{\tilde{z}}, \underline{\tilde{\tilde{z}}})$ defined as solutions of the two *linear* systems (4.3a) respectively (4.3b) can be expressed more explicitly than via the standard Cramer formula (ratio of two determinants).

Clearly the 4 models treated in this paper belong to this class (4.1) (the last one, however, only if $\rho = 0$): see (3.6a), (3.18), (3.22a) and (3.31). The interest of additional models treatable via this approach depends on the neatness of the corresponding equations of motion, which can only be investigated on a case-by-case basis.

A Appendix

In this appendix we collect various mathematical developments whose treatment in the body of the paper would interrupt the flow of the presentation.

First of all we report several mathematical identities. We consider all of them well-known, but for completeness we either prove them below, or indicate where proofs can be found. These formulas feature sets of N numbers such as ξ_n or η_n or ζ_n ; these numbers are *arbitrary* but for simplicity we assume them to be *distinct*. The formulas of course remain valid when these numbers are not distinct, but possibly only by taking appropriate limits. Sometimes an *arbitrary* number z also appears.

$$\sum_{k=1}^N \left[\prod_{j=1, j \neq k}^N \left(\frac{\zeta_j - z}{\zeta_j - \zeta_k} \right) \right] = 1, \quad (\text{A.1})$$

$$\sum_{k=1}^N \left[\frac{1}{\zeta_k} \prod_{j=1, j \neq k}^N \left(\frac{1}{\zeta_j - \zeta_k} \right) \right] = \prod_{j=1}^N \left(\frac{1}{\zeta_j} \right), \quad (\text{A.2})$$

$$\sum_{k=1}^N \left[\zeta_k^{n-1} \prod_{j=1, j \neq k}^N \left(\frac{\zeta_j - z}{\zeta_j - \zeta_k} \right) \right] = z^{n-1}, \quad n = 1, 2, \dots, N, \quad (\text{A.3})$$

$$\sum_{k=1}^N \left[\zeta_k^{n-1} \prod_{j=1, j \neq k}^N \left(\frac{1}{\zeta_k - \zeta_j} \right) \right] = \delta_{nN}, \quad n = 1, 2, \dots, N, \quad (\text{A.4})$$

$$\prod_{j=1, j \neq m}^N \left(\frac{\zeta_j - \zeta_n}{\zeta_j - \zeta_m} \right) = \delta_{nm}, \quad n, m = 1, 2, \dots, N, \quad (\text{A.5})$$

$$\sum_{k=1}^N \left\{ \left[\prod_{j=1, j \neq k}^N \left(\frac{\xi_j - \eta_n}{\xi_j - \xi_k} \right) \right] \left[\prod_{j=1, j \neq m}^N \left(\frac{\eta_j - \xi_k}{\eta_j - \eta_m} \right) \right] \right\} = \delta_{nm}, \quad n, m = 1, 2, \dots, N, \quad (\text{A.6})$$

$$\sum_{k=1}^N \left[\left(\frac{1}{z - \eta_k} \right) \frac{\prod_{j=1, j \neq n}^N (\zeta_j - \eta_k)}{\prod_{j=1, j \neq k}^N (\eta_j - \eta_k)} \right] = \frac{\prod_{j=1, j \neq n}^N (\zeta_j - z)}{\prod_{j=1}^N (\eta_j - z)} \equiv \frac{1}{z - \zeta_n} \prod_{j=1}^N \left(\frac{\zeta_j - z}{\eta_j - z} \right), \quad n = 1, 2, \dots, N, \quad (\text{A.7})$$

$$\sum_{k=1}^N \left[\frac{\prod_{j=1, j \neq n}^N (\zeta_j - \eta_k)}{\prod_{j=1, j \neq k}^N (\eta_j - \eta_k)} \right] \equiv \sum_{k=1}^N \left[\left(\frac{\eta_k - \zeta_k}{\eta_k - \zeta_n} \right) \prod_{j=1, j \neq k}^N \left(\frac{\zeta_j - \eta_k}{\eta_j - \eta_k} \right) \right] = 1, \quad n = 1, 2, \dots, N, \quad (\text{A.8})$$

$$\sum_{k=1}^N \left[\left(\frac{1}{\eta_k - \xi_n} \right) \prod_{j=1, j \neq k}^N \left(\frac{1}{\eta_j - \eta_k} \right) \right] = \prod_{j=1}^N \left(\frac{1}{\eta_j - \xi_n} \right), \quad n = 1, 2, \dots, N, \quad (\text{A.9})$$

$$\begin{aligned} \sum_{k=1}^N \left[\left(\frac{\eta_k - \zeta_k}{\eta_k - z} \right) \prod_{j=1, j \neq k}^N \left(\frac{\zeta_j - \eta_k}{\eta_j - \eta_k} \right) \right] &\equiv \sum_{k=1}^N \left[\frac{\prod_{j=1}^N (\zeta_j - \eta_k)}{(z - \eta_k) \prod_{j=1, j \neq k}^N (\eta_j - \eta_k)} \right] \\ &= 1 - \prod_{j=1}^N \left(\frac{\zeta_j - z}{\eta_j - z} \right), \end{aligned} \quad (\text{A.10})$$

$$\sum_{k=1}^N \left[\frac{\prod_{j=1}^N (\zeta_j - \eta_k)}{\prod_{j=1, j \neq k}^N (\eta_j - \eta_k)} \right] = \sum_{k=1}^N (\zeta_k - \eta_k), \quad (\text{A.11})$$

$$\sum_{k=1}^N \left[\frac{\eta_k \prod_{j=1, j \neq n}^N (\zeta_j - \eta_k)}{\prod_{j=1, j \neq k}^N (\eta_j - \eta_k)} \right] = \sum_{k=1, k \neq n}^N (\zeta_k) - \sum_{k=1}^N (\eta_k), \quad n = 1, \dots, N. \quad (\text{A.12})$$

The identity (A.1) (with z an arbitrary number) is implied by the fact that its left-hand side is a polynomial in z of degree less than N (in fact, of degree at most $N - 1$) which clearly has the value *unity* at the N points ζ_n , and the right-hand side, i.e. *unity*, is the *unique* polynomial of degree less than N in z that has the value *unity* in N distinct points. The identity (A.2) is the special case of (A.1) with $z = 0$. The identity (A.3) coincides with equation (2.4.2-32) of [4] (or, as above, it is implied by the observation that its left-hand side is a polynomial in z of degree less than N the values of which at the N points ζ_k coincide with the values of the polynomial z^n at $z = \zeta_k$). The identity (A.4) coincides with equations (2.4.3-12) and (2.4.3-21) of [4]. The identity (A.5) is obvious. The identities (A.6) respectively (A.7) coincide with equations (2.4.2-26) respectively (2.4.2-27) of [4] (via the definition (2.4.2-24), with $x_n = \xi_n$, $y_n = \eta_n$, respectively $x_n = z$, $y_n = \eta_n$, $z_n = \zeta_n$). The identities (A.8) respectively (A.9) follow from (A.7) in the limit $z \rightarrow \infty$ respectively $\zeta_j \rightarrow \infty$. The identity (A.10) follows from (A.8) and (A.7) via the trivial identity

$$\frac{\zeta_n - \eta_k}{z - \eta_k} \equiv 1 - \frac{\zeta_n - z}{\eta_k - z}.$$

Finally, the identity (A.11) follows from (A.10) in the limit $z \rightarrow \infty$, and the identity (A.12) is just the special case of the preceding identity (A.11) with $\zeta_n = 0$.

Next we report a simple lemma (for a neat proof see for instance [18], or below, after the proof of the following Lemma A.2).

Lemma A.1. *The solution of the set of N linear algebraic equations for the N variables g_k reading*

$$\sum_{k=1}^N \frac{g_k}{\xi_k - \eta_n} = c, \quad n = 1, \dots, N \quad (\text{A.13a})$$

is provided by the formula

$$g_k = c(\xi_k - \eta_k) \prod_{j=1, j \neq k}^N \left(\frac{\xi_k - \eta_j}{\xi_k - \xi_j} \right), \quad k = 1, \dots, N. \quad (\text{A.13b})$$

A generalization of this lemma reads as follows:

Lemma A.2. *The solution of the set of N linear algebraic equations for the N variables g_k reading*

$$\sum_{k=1}^N \frac{g_k}{\xi_k - \eta_n} = c_n, \quad n = 1, \dots, N \quad (\text{A.14a})$$

is provided by the formula

$$g_k = \sum_{s=1}^N \left\{ c_s(\xi_k - \eta_s) \left[\prod_{j=1, j \neq k}^N \left(\frac{\xi_j - \eta_s}{\xi_j - \xi_k} \right) \right] \left[\prod_{j=1, j \neq s}^N \left(\frac{\eta_j - \xi_k}{\eta_j - \eta_s} \right) \right] \right\},$$

$$n = 1, \dots, N, \quad (\text{A.14b})$$

or equivalently

$$g_k = (\xi_k - \eta_k) \left[\prod_{j=1, j \neq k}^N \left(\frac{\eta_j - \xi_k}{\xi_j - \xi_k} \right) \right] \sum_{s=1}^N \left\{ c_s \left[\frac{\prod_{j=1, j \neq k}^N (\xi_j - \eta_s)}{\prod_{j=1, j \neq s}^N (\eta_j - \eta_s)} \right] \right\},$$

$$n = 1, \dots, N. \quad (\text{A.14c})$$

To prove this formula one inserts this expression, (A.14b), of g_k in (A.14a), and notes that one obtains thereby an equality provided there holds the formula

$$\sum_{k=1}^N \left\{ \frac{\xi_k - \eta_s}{\xi_k - \eta_n} \left[\prod_{j=1, j \neq k}^N \left(\frac{\xi_j - \eta_s}{\xi_j - \xi_k} \right) \right] \left[\prod_{j=1, j \neq s}^N \left(\frac{\eta_j - \xi_k}{\eta_j - \eta_s} \right) \right] \right\} = \delta_{sn}$$

or, equivalently,

$$\sum_{k=1}^N \left\{ \left[\prod_{j=1, j \neq k}^N \left(\frac{\xi_j - \eta_n}{\xi_j - \xi_k} \right) \right] \left[\prod_{j=1, j \neq s}^N \left(\frac{\eta_j - \xi_k}{\eta_j - \eta_s} \right) \right] \right\} = \delta_{sn} \prod_{j=1}^N \left(\frac{\xi_j - \eta_n}{\xi_j - \eta_s} \right).$$

Clearly this formula is implied by the identity (A.6). Lemma A.2 is thus proven.

Note that, by setting $c_n = c$ in (A.14c) and using the identity (A.8) (with the dummy index k replaced by s and the index n replaced by k), one reobtains (A.13b), thereby proving Lemma A.1.

We now report, and prove, 3 other lemmata.

Lemma A.3. *There holds the formula*

$$\det[I + X] = 1 + \sum_{k=1}^N x_k, \quad (\text{A.15a})$$

provided I is the $N \times N$ unit matrix and the $N \times N$ matrix X is defined componentwise as follows:

$$X_{nm} = x_m, \quad n, m = 1, \dots, N. \quad (\text{A.15b})$$

This (presumably well-known) formula is easily proven by recursion.

Lemma A.4. *The N eigenvalues of the $N \times N$ matrix*

$$U_{nm} = \delta_{nm}\zeta_n + \eta_m, \quad n, m = 1, \dots, N, \quad (\text{A.16a})$$

coincide with the N solutions of the following algebraic equation in z :

$$\sum_{k=1}^N \left(\frac{\eta_k}{z - \zeta_k} \right) - 1 = 0, \quad (\text{A.16b})$$

i.e. they are the N roots of the polynomial of degree N in z that obtains by multiplying the left-hand side of this equation by $\prod_{j=1}^N (z - \zeta_j)$.

This lemma is an immediate consequence of the preceding Lemma A.3, because the secular equation associated with the matrix (A.16a) (whose roots provide the eigenvalues) is easily seen to coincide (up to an overall, hence irrelevant, multiplicative constant) with the vanishing of the determinant in the left-hand side of (A.15a) with (A.15b) and $x_k = \eta_k/(z - \zeta_k)$.

Lemma A.5. *The inverse of the matrix defined componentwise as follows,*

$$M_{nm} = \frac{f_n g_m}{\xi_m - \eta_n}, \quad n, m = 1, \dots, N, \quad (\text{A.17a})$$

is defined componentwise as follows:

$$[M^{-1}]_{nm} = \left(\frac{\xi_n - \eta_m}{g_n f_m} \right) \left[\prod_{j=1, j \neq n}^N \left(\frac{\xi_j - \eta_m}{\xi_j - \xi_n} \right) \right] \left[\prod_{j=1, j \neq m}^N \left(\frac{\eta_j - \xi_n}{\eta_j - \eta_m} \right) \right],$$

$$n, m = 1, \dots, N. \quad (\text{A.17b})$$

The proof of this formula goes as follows. The matrix formula $MM^{-1} = I$, written componentwise, reads, via (A.17a),

$$\sum_{k=1}^N \left(\frac{g_k [M^{-1}]_{km}}{\xi_k - \eta_n} \right) = \frac{\delta_{nm}}{f_n}, \quad n, m = 1, \dots, N.$$

Then, for fixed m , apply Lemma A.2 with g_k replaced by $g_k [M^{-1}]_{km}$, and c_n replaced by δ_{nm}/f_n . This yields, rather immediately, the formula (A.17b), which is thereby proven.

B Appendix

In this appendix we detail the derivation of some findings for the fourth model, firstly the expression (2.17b) of $\hat{g}_n(\underline{z}, \underline{\tilde{z}})$ as solution of the linear system (3.35b), and secondly the derivation of (2.19b) from (2.19a).

Via the identity

$$\frac{1}{(\eta\tilde{z}_k + \beta)(\tilde{z}_k - az_n - b)} = \frac{1}{\eta(az_n + b) + \beta} \left(\frac{1}{\tilde{z}_k - az_n - b} - \frac{\eta}{\eta\tilde{z}_k + \beta} \right), \quad (\text{B.1})$$

and the definition

$$\sigma^{(0)} = \sum_{k=1}^N \left(\frac{\eta g_k}{\eta\tilde{z}_k + \beta} \right), \quad (\text{B.2})$$

the linear system (3.35b) can be conveniently reformulated as follows:

$$\sum_{k=1}^N \left(\frac{g_k}{\tilde{z}_k - az_n - b} \right) = c_n, \\ c_n = \sigma^{(0)} + \frac{\eta(az_n + b) + \beta}{\eta z_n + \beta} = \sigma^{(0)} + a + \frac{a}{\eta} \left[\frac{\beta(1 - \alpha)}{az_n + a\beta/\eta} \right].$$

We now use Lemma A.2 (with $\xi_k = \tilde{z}_k$, $\eta_n = a z_n + b$ and c_n defined as above). We thus get

$$\hat{g}_n(\underline{z}, \underline{\tilde{z}}) = (\tilde{z}_n - az_n - b) \left[\prod_{j=1, j \neq n}^N \left(\frac{\tilde{z}_n - az_j - b}{\tilde{z}_n - \tilde{z}_j} \right) \right] \left[(\sigma_0 + a)\sigma_n^{(1)} + \frac{a\beta(1 - \alpha)}{\eta} \sigma_n^{(2)} \right], \\ \sigma_n^{(1)} = \sum_{k=1}^N \left[\frac{\prod_{j=1, j \neq n}^N (\tilde{z}_j - az_k - b)}{\prod_{j=1, j=k}^N (az_j - az_k)} \right], \quad n = 1, \dots, N, \\ \sigma_n^{(2)} = \sum_{k=1}^N \left[\frac{1}{az_k + a\beta/\eta} \frac{\prod_{j=1, j \neq n}^N (\tilde{z}_j - az_k - b)}{\prod_{j=1, j=k}^N (az_j - az_k)} \right], \quad n = 1, \dots, N.$$

It is now plain, via the identity (A.8) (with $\eta_k = az_k + b$, $\zeta_j = \tilde{z}_j$) that $\sigma_n^{(1)} = 1$. As for the sum $\sigma_n^{(2)}$, it is also easily evaluated via the identity (A.7) (now with $\eta_k = az_k + b$, $\zeta_j = \tilde{z}_j$, $z = b - a\beta/\eta$):

$$\sigma_n^{(2)} = \frac{a^{-N}\eta}{\eta\tilde{z}_n + a\beta - b\eta} \prod_{j=1}^N \left(\frac{\eta\tilde{z}_j + a\beta - b\eta}{\eta z_j + \beta} \right).$$

Hence

$$\hat{g}_n(\underline{z}, \underline{\tilde{z}}) = (\tilde{z}_n - az_n - b) \left[\prod_{j=1, j \neq n}^N \left(\frac{az_j + b - \tilde{z}_n}{\tilde{z}_j - \tilde{z}_n} \right) \right] \left(\sigma^{(0)} + a + \frac{\sigma^{(3)}}{\eta\tilde{z}_n + a\beta - b\eta} \right), \quad (\text{B.3a})$$

$$\sigma^{(3)} = a^{1-N}\beta(1 - \alpha) \prod_{j=1}^N \left(\frac{\eta\tilde{z}_j + \alpha\beta}{\eta z_j + \beta} \right). \quad (\text{B.3b})$$

Now, using this expression of $\hat{g}_n(\underline{z}, \underline{\tilde{z}})$, we can evaluate σ_0 from its definition (B.2), thereby obtaining

$$\begin{aligned}\sigma^{(0)} &= \frac{a\sigma^{(4)} + \sigma^{(3)}\sigma^{(5)}}{1 - \sigma^{(4)}} = -a + \frac{a + \sigma^{(3)}\sigma^{(5)}}{1 - \sigma^{(4)}}, \\ \sigma^{(4)} &= \sum_{k=1}^N \left[\left(\frac{\tilde{z}_k - az_k - b}{\tilde{z}_k + \beta/\eta} \right) \prod_{j=1, j \neq k}^N \left(\frac{\tilde{z}_k - az_j - b}{\tilde{z}_k - \tilde{z}_j} \right) \right], \\ \sigma^{(5)} &= \sum_{k=1}^N \left[\frac{\tilde{z}_k - az_k - b}{(\eta\tilde{z}_k + \beta)(\tilde{z}_k + \alpha\beta/\eta)} \prod_{j=1, j \neq k}^N \left(\frac{az_j + b - \tilde{z}_k}{\tilde{z}_j - \tilde{z}_k} \right) \right].\end{aligned}$$

It is now plain (via (A.10) with $\eta_j = \tilde{z}_j$, $\zeta_j = az_j + b$, $z = -\beta/\eta$) that

$$\sigma^{(4)} = 1 - \prod_{j=1}^N \left(\frac{\eta az_j + \eta b + \beta}{\eta \tilde{z}_j + \beta} \right). \quad (\text{B.4})$$

To evaluate σ_5 , we use again an identity analogous to that used above:

$$\frac{1}{(\eta\tilde{z}_k + \beta)(\tilde{z}_k + \alpha\beta/\eta)} = \frac{1}{\beta(1 - \alpha)} \left(\frac{1}{\tilde{z}_k + \alpha\beta/\eta} - \frac{\eta}{\eta\tilde{z}_k + \beta} \right).$$

Thereby

$$\sigma^{(5)} = \frac{\sigma^{(6)} - \sigma^{(7)}}{\beta(1 - \alpha)}, \quad (\text{B.5a})$$

$$\sigma^{(6)} = \sum_{k=1}^N \left[\frac{\tilde{z}_k - az_k - b}{\tilde{z}_k + \alpha\beta/\eta} \prod_{j=1, j \neq k}^N \left(\frac{az_j + b - \tilde{z}_k}{\tilde{z}_j - \tilde{z}_k} \right) \right], \quad (\text{B.5b})$$

$$\sigma^{(7)} = \sum_{k=1}^N \left[\frac{\tilde{z}_k - az_k - b}{\tilde{z}_k + \beta/\eta} \prod_{j=1, j \neq k}^N \left(\frac{az_j + b - \tilde{z}_k}{\tilde{z}_j - \tilde{z}_k} \right) \right]. \quad (\text{B.5c})$$

Both these sums can be evaluated via the identity (A.10), with $\eta_k = \tilde{z}_k$, $\zeta_j = az_j + b$, and with $z = \alpha\beta/\eta$ respectively with $z = -\beta/\eta$, obtaining

$$\sigma^{(6)} = 1 - a^N \prod_{j=1}^N \left[\frac{\eta z_j + \beta}{\eta \tilde{z}_j + \alpha\beta} \right], \quad \sigma^{(7)} = 1 - \prod_{j=1}^N \left[\frac{\eta(az_j + b) + \beta}{\eta \tilde{z}_j + \beta} \right],$$

hence (via (B.5a))

$$\sigma^{(5)} = \frac{1}{\beta(1 - a) + b\eta} \left\{ \prod_{j=1}^N \left[\frac{\eta(az_j + b) + \beta}{\eta \tilde{z}_j + \beta} \right] - a^N \prod_{j=1}^N \left[\frac{\eta z_j + \beta}{\eta \tilde{z}_j + \alpha\beta} \right] \right\},$$

and via this formula together with (B.4) and (B.3b) we finally get

$$\sigma^{(0)} + a = a^{1-N} \prod_{j=1}^N \left(\frac{\eta \tilde{z}_j + \alpha\beta}{\eta z_j + \beta} \right).$$

The insertion of this expression of $\sigma^{(0)} + a$ and of the expression (B.3b) of $\sigma^{(3)}$ in (B.3a) completes the derivation of the expression (2.17b) of $\hat{g}_n(\underline{z}, \underline{\tilde{z}})$.

Finally let us tersely outline the derivation of (2.19b) from (2.19a). Firstly one uses in (2.19a) the identity (B.1); then one uses twice the identity (A.10), with $\zeta_k = \tilde{z}_k - b$, $\eta_k = a\tilde{z}_k$ and with $z = a(az_n + b)$ respectively with $z = -a\beta/\eta$. And the rest is trivial algebra.

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